Lecture 2

Quasi-geostrophic waves and transport

(i) Quasigeostrophic equations and potential vorticity

(ii) Wave activity conservation

(iii) Stability of zonal flows

(iv) PV transport and nonacceleration

(v) Mean momentum and heat budgets

(vi) Rossby waves: barotropic, baroclinic, and breaking

FDEPS 2010
Alan Plumb, MIT
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(i) Quasigeostrophic equations and potential vorticity

Hydrostatic equations with rotation
(log-p coordinates, $f = 2\Omega \sin\varphi$):

$$\frac{\partial u}{\partial t} + \mathbf{u.}\nabla u - fv = -\frac{\partial \phi}{\partial x} + G^{(x)}$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + fu = -\frac{\partial \phi}{\partial y} + G^{(y)}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = (\rho \Pi)^{-1} J$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial \phi}{\partial z} - \frac{\kappa \Pi}{H}\theta = 0$$

Assumptions:

- Midlatitude “beta-plane” $f = f_0 + \beta y$

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \simeq 0$$

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- $\beta L/f_0 \ll 1$ → geostrophic flow nondivergent → $w \simeq 0$
- At leading order $\partial\theta/\partial z$ is function of z only (for consistent entropy budget)

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Define background state

$$\Theta_0(z)\ ,\quad \Phi_0(z) = \frac{\kappa}{H}\int_0^z \Pi\Theta_0\, dz$$

Geostrophic flow:

$$-fv_g = -\frac{\partial\phi}{\partial x}\ ;\quad +fu = -\frac{\partial\phi}{\partial y}$$

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$

$$\boxed{u_g = -\frac{\partial\psi}{\partial y}\ ;\ v_g = \frac{\partial\psi}{\partial x}\ ;\ w_g = 0}$$

geostrophic streamfunction:

$$\psi = [\phi - \Phi_0(z)]/f_0$$

Hydrostatic balance

$$\frac{\partial\psi}{\partial z} = \frac{\kappa\Pi}{f_0 H}[\theta - \Theta_0(z)]$$

→ thermal wind shear

$$\boxed{f_0\frac{\partial u}{\partial z} = -\frac{\kappa\Pi}{f_0 H}\frac{\partial\theta}{\partial y}\ ;\ f_0\frac{\partial v}{\partial z} = \frac{\kappa\Pi}{f_0 H}\frac{\partial\theta}{\partial x}}$$

Quasi-geostrophic equations 2

At next order,

$$D_g u_g - \beta y v_g - f_0 v_a = G^{(x)} \tag{1}$$

$$D_g v_g + \beta y u_g + f_0 u_a = G^{(y)} \tag{2}$$

$$D_g \theta + w_a \frac{\partial \Theta_0}{\partial z} = (\rho \Pi)^{-1} J \tag{3}$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho w_a) = 0$$

where D_g is derivative following *geostrophic* flow:

$$D_g \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

and (u_a, v_a, w_a) is the *ageostrophic* velocity

$$(u_a, v_a, w_a) = (u - u_g, v - v_g, w)$$

From these, we can derive
$\{\partial(2)/\partial x - \partial(1)/\partial y + (f_0/\rho)\partial(\rho \times [3]/\Theta_{0,z})/\partial z\}$
the equation for *quasigeostrophic potential vorticity*, q :

$$\rightarrow \quad \boxed{D_g q = X}$$

where

$$\begin{aligned} q &= f_0 + \beta y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{f_0}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{\tilde{\theta}}{\Theta_{0,z}}\right) \\ &= f_0 + \beta y + \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\psi \end{aligned}$$

and

$$X = \frac{\partial \mathcal{G}^{(y)}}{\partial x} - \frac{\partial \mathcal{G}^{(x)}}{\partial y} + \frac{f_0}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{J}{\Pi \Theta_{0,z}}\right)$$

→ for conservative flow ($\mathbf{G} = 0$, $J = 0$, whence $X = 0$):
q is conserved *following the geostrophic flow*.

(ii) Wave activity conservation

PV fluxes and the Eliassen-Palm theorem

Consider *small-amplitude* motions on a steady, zonally-uniform *basic state*

$$[u_g, v_g, w] = [U(y,z), 0, 0] \; ; \; \theta = \Theta(y,z); \; \psi = \Psi(y,z) \; ; \; Q(y,z)$$

where

$$\frac{\partial \Psi}{\partial y} = -U \; ; \qquad \frac{\kappa \Pi}{H} \frac{\partial \theta}{\partial y} = -f_\circ \frac{\partial U}{\partial z}$$

$$Q(y,z) = f_0 + \beta y + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\rho f_0^2}{N^2} \frac{\partial \Psi}{\partial z} \right)$$

Write

$$\psi = \Psi + \psi'(x,y,z,t)$$

then $v' = \partial \psi' / \partial x$ and

$$q' = \Delta^2 \psi' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right) \; .$$

so PV flux is

$$\overline{v'q'} = \overline{\frac{\partial \psi'}{\partial x} \left[\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right) \right]}$$

Consider $\overline{v'q'}$:

$$\text{(I)}\quad \overline{\frac{\partial\psi'}{\partial x}\frac{\partial^2\psi'}{\partial x^2}} = \frac{1}{2}\overline{\frac{\partial}{\partial x}\left[\left(\frac{\partial\psi'}{\partial x}\right)^2\right]} = 0\ ;$$

$$\text{(II)}\quad \overline{\frac{\partial\psi'}{\partial x}\frac{\partial^2\psi'}{\partial y^2}} = \overline{\frac{\partial}{\partial y}\left[\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y}\right]} - \overline{\frac{\partial\psi'}{\partial y}\frac{\partial^2\psi'}{\partial x\partial y}}$$

$$= \overline{\frac{\partial}{\partial y}\left[\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y}\right]} - \frac{1}{2}\overline{\frac{\partial}{\partial x}\left[\left(\frac{\partial\psi'}{\partial y}\right)^2\right]}$$

$$= \frac{\partial}{\partial y}\left(\overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y}}\right);$$

$$\text{(III)}\quad \overline{\frac{\partial\psi'}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial z}\left[\frac{\rho f_0^2}{N^2}\frac{\partial\psi'}{\partial z}\right]} = \frac{1}{\rho}\overline{\frac{\partial}{\partial z}\left[\frac{\rho f_0^2}{N^2}\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z}\right]} - \frac{f_0^2}{N^2}\overline{\frac{\partial\psi'}{\partial z}\frac{\partial^2\psi'}{\partial x\partial z}}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial z}\left[\frac{\rho f_0^2}{N^2}\overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z}}\right] - \frac{f_0^2}{2N^2}\overline{\frac{\partial}{\partial x}\left[\left(\frac{\partial\psi'}{\partial z}\right)^2\right]}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial z}\left[\frac{\rho f_0^2}{N^2}\overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z}}\right].$$

Therefore

$$\rho\overline{v'q'} = \nabla \bullet \mathbf{F}$$

where

$$\begin{aligned}\mathbf{F} &= (F^{(y)}, F^{(z)}) \\ &= \left(\rho\overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y}} \;,\; \frac{\rho f_0^2}{N^2}\overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z}} \right) \\ &= \left(-\rho\overline{u'v'} \;,\; \rho f_0 \frac{\overline{v'\theta'}}{d\Theta_0/dz} \right)\end{aligned}$$

F is known as the ELIASSEN-PALM flux.

Linearizing the QGPV equation:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)q' + v'\frac{\partial Q}{\partial y} = X'$$

multiply by q' and average:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{q'^2}\right) + \overline{v'q'}\frac{\partial Q}{\partial y} = \overline{v'X'}$$

Define

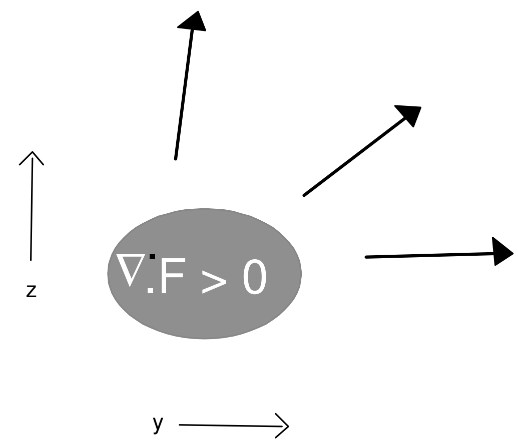
$$A = \rho\frac{1}{2}\overline{q'^2}/\left(\frac{\partial Q}{\partial y}\right) \text{ and } \qquad \mathcal{D} = \rho\overline{v'X'}/\left(\frac{\partial Q}{\partial y}\right),$$

then

$$\boxed{\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = \mathcal{D}}$$

→ the ELIASSEN-PALM RELATION:

— a conservation law for zonally-averaged *wave activity* whose density is A.. Note that $\mathcal{D} \to 0$ for conservative flow.



F is a meaningful measure of the *propagation* of wave activity

The Eliassen-Palm theorem

For steady ($\partial A/\partial t = 0$), small amplitude, conservative ($\mathcal{D} = 0$) waves:

$$\nabla \cdot \mathbf{F} = 0 \quad : \quad \rho\overline{v'q'} = 0$$

(iii) Stability of zonal flows

Stability of zonal flows to QG perturbations: The Charney-Stern theorem
Charney & Stern, *J. Atmos. Sci.*, **19**, 159-172, (1962)

Integrate the EP relation:

$$\frac{\partial}{\partial t}\iint_{\mathcal{R}} A\ dy\ dz + \oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\ dl = \iint_{\mathcal{R}} \mathcal{D}\ dy\ dz$$

over the domain $\mathcal{R}$ bounded by the surface.
Boundary fluxes:

at sides $y = y_1, y_2, \quad v = 0$:

$$\rightarrow\ \mathbf{F}\cdot\mathbf{n} = F^{(y)} = -\rho\overline{u'v'} = 0$$

at top and bottom:

$$\mathbf{F}\cdot\mathbf{n} = F^{(z)} = \rho f_0 \frac{\overline{v'\theta'}}{d\Theta_0/dz}$$

which is *nonzero* if $\overline{v'\theta'} \neq 0$. But if the upper and lower boundaries are isentropic, then

$$\theta' = 0 \rightarrow \mathbf{F}\cdot\mathbf{n} = 0$$

there.

Hence for

(i) conservative flow (no creation or dissipation of wave activity)

(ii) with isentropic upper and lower boundaries
(no flux through boundaries)

$$\frac{\partial}{\partial t} \iint_{\mathcal{R}} A \; dy \; dz = 0$$

→ globally integrated wave activity is conserved.

But sign of A depends on sign of $\partial\bar{q}/\partial y$:

$$A = \frac{\frac{1}{2}\rho\overline{q'^2}}{\partial\bar{q}/\partial y}$$

Look for *normal mode* growth such that $\overline{q'^2} = B(t)C(y, z)$
(both B and C positive definite)

$$\frac{dB}{dt} \iint_{\mathcal{R}} \frac{1}{2} \frac{C(y, z)}{\partial\bar{q}/\partial y} \; dy \; dz = 0$$

If mean PV gradient is single-signed, $dB/dt = 0$ → no growth

Hence

A zonal flow is *stable* to inviscid, adiabatic, quasigeostrophic normal mode perturbations if

a. there is no change of sign of PV gradient within the fluid and

b. the system is bounded above and below by isentropic boundaries.

The *Charney-Stern theorem*. (does not apply to non-normal-mode growth).

(iv) PV transport and nonacceleration

Potential vorticity transport and the nonacceleration theorem

How do eddies influence the zonal mean circulation?
Take mean of QGPV equation

$$\frac{\partial \bar{q}}{\partial t} + \frac{\partial}{\partial y}(\overline{v'q'}) = \bar{X} .$$

Note (i) $v_g = \overline{\partial\psi/\partial x} = 0$, so no mean advection
(ii) $w_g = 0$, so no vertical eddy flux to leading order
→influence of eddies described entirely by the northward flux $\overline{v'q'} = \rho^{-1}\nabla \cdot \mathbf{F}$

Know from the Eliassen-Palm theorem that if the waves are everywhere

(I) of small amplitude,

(II) conservative, and

(III) statistically steady

→ **F** is nondivergent and $\overline{v'q'} = 0$. Then $\partial\bar{q}/\partial t$ is *independent of the waves* (if we assume that $\bar{X}$ is also independent).

Then $\partial\bar{q}/\partial t$ is *independent of the waves*
(if we assume that $\bar{X}$ is also independent). Now,

$$\bar{q} = f + \Delta^2(\bar{\psi})$$

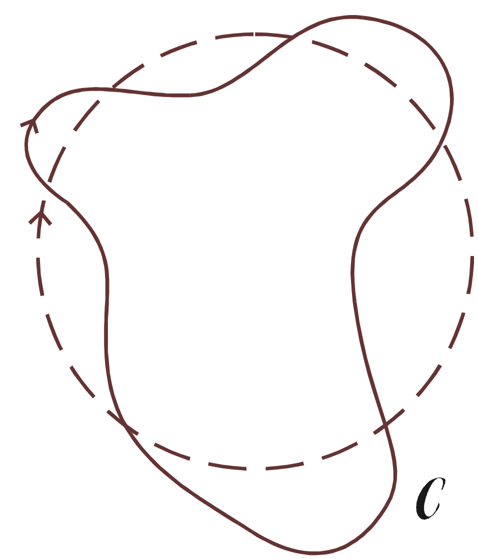
therefore can invert PV:

$$\frac{\partial\bar{\psi}}{\partial t} = \Delta^{-2}\frac{\partial\bar{q}}{\partial t} = \Delta^{-2}\bar{X}$$

Δ^2 is an elliptic operator, so solution invokes boundary conditions on $\partial\bar{\psi}/\partial t$.
If we invoke the further condition that

(IV) the boundary conditions on $\partial\bar{\psi}/\partial t$ are independent of the waves

then $\partial\bar{\psi}/\partial t$ is everywhere independent of the waves.
$\bar{u} = -\partial\bar{\psi}/\partial y$, $\bar{\theta} = (f_0 H/\kappa\Pi)\partial\bar{\psi}/\partial z$ → same true of $\partial\bar{u}/\partial t$, $\partial\bar{\theta}/\partial t$.
→ *nonacceleration theorem* (Charney-Drazin, Andrews-McIntyre)
Closely related to Kelvin's circulation theorem:



(v) Mean momentum and heat budgets

Mean momentum and heat budgets

Zonal mean QG eqs:

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_a &= \overline{G^{(x)}} - \frac{\partial}{\partial y}(\overline{u'v'}) \\
\frac{\partial \bar{\theta}}{\partial t} + \overline{w}_a \frac{\partial \bar{\theta}}{\partial z} &= (\rho \Pi)\bar{J} - \frac{\partial}{\partial y}(\overline{v'\theta'}) \\
\frac{\partial \bar{v}_a}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho \overline{w}_a) &= 0 \\
f_0 \frac{\partial \bar{u}}{\partial z} + \frac{\kappa \Pi}{f_0 H}\frac{\partial \bar{\theta}}{\partial y} &= 0
\end{aligned}
$$

set of 4 equations in the 4 unknowns $\partial \overline{u}/\partial t$, $\partial \overline{T}/\partial t$, $\bar{v}_a$ and $\bar{w}_a$
in terms of the two eddy driving terms $\overline{u'v'}$, $\overline{v'\theta'}$

Central role of the PV flux—obvious in mean PV budget—not obvious here

Transformed Eulerian-mean theory
(Andrews & McIntyre, *J. Atmos. Sci.*, 1977; Andrews et al., 1981)

Define ageostrophic "residual" mean streamfunction

$$(\bar{v}_*, \bar{w}_*) = \left[\bar{v}_a - \frac{1}{\rho}\frac{\partial(\rho\chi_*)}{\partial z}, \bar{w}_a + \frac{\partial \chi_*}{\partial y} \right]$$

where

$$\chi_* = \frac{\overline{v'\theta'}}{\partial\bar{\theta}/\partial z}$$

(and remember $\bar{\theta} = \bar{\theta}(z)$ to leading order). Then

$$\begin{aligned}
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_a &= \overline{G^{(x)}} - \frac{\partial}{\partial y}(\overline{u'v'}) \\
\frac{\partial \bar{\theta}}{\partial t} + \bar{w}_a \frac{\partial \bar{\theta}}{\partial z} &= (\rho\Pi)\bar{J} - \frac{\partial}{\partial y}(\overline{v'\theta'}) \\
\frac{\partial \bar{v}_a}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho \bar{w}_a) &= 0 \\
f_0 \frac{\partial \bar{u}}{\partial z} + \frac{\kappa \Pi}{f_0 H}\frac{\partial \bar{\theta}}{\partial y} &= 0
\end{aligned}$$

$\longrightarrow$

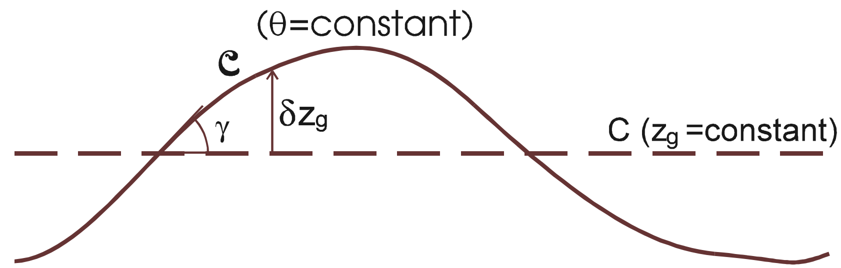
$$\begin{aligned}
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_* &= \overline{G^{(x)}} + \frac{1}{\rho}\nabla \cdot \mathbf{F} \\
\frac{\partial \bar{\theta}}{\partial t} + \bar{w}_* \frac{\partial \bar{\theta}}{\partial z} &= (\rho\Pi)^{-1}\bar{J} \\
\frac{\partial \bar{v}_*}{\partial y} + \frac{1}{\rho}\frac{\partial(\rho \bar{w}_*)}{\partial z} &= 0 \\
f_0 \frac{\partial \bar{u}}{\partial z} + \frac{\kappa \Pi}{f_0 H}\frac{\partial \bar{\theta}}{\partial y} &= 0
\end{aligned}$$

where **F** is the EP flux, as before.
Now have set of equations for $\bar{v}_*, \bar{w}_*, \partial\bar{u}/\partial t$ and $\partial\bar{T}/\partial t$ in terms of *one* eddy forcing term $\rho^{-1}\nabla \cdot \mathbf{F} = \overline{v'q'}$, appearing as effective *body force* (per unit mass)
Nonacceleration theroem then follows directly.

$$
\begin{aligned}
\mathbf{F} &= \left(F^{(y)}, F^{(z)}\right) \\
&= \left(\rho \overline{\frac{\partial \psi'}{\partial x}\frac{\partial \psi'}{\partial y}}\ ,\ \frac{\rho f_0^2}{N^2}\overline{\frac{\partial \psi'}{\partial x}\frac{\partial \psi'}{\partial z}} \right) \\
&= \left(-\rho \overline{u'v'}\ ,\ \rho f_0 \frac{\overline{v'\theta'}}{d\Theta_0/dz} \right)
\end{aligned}
$$

F as a momentum flux:

Consider adiabatic flow; isentropic surface $\mathcal{C}$ (of constant θ) , disturbed by small-amplitude waves.
Zonally-averaged zonal stress on $\mathcal{C}$ is τ where



$$\rho\tau = -\overline{p\sin\gamma} \simeq -\overline{p\gamma} \simeq -\overline{\gamma\ \delta p}$$

(since $\bar{\gamma} = 0$) where δp is the pressure variation *along* $\mathcal{C}$.
γ is small

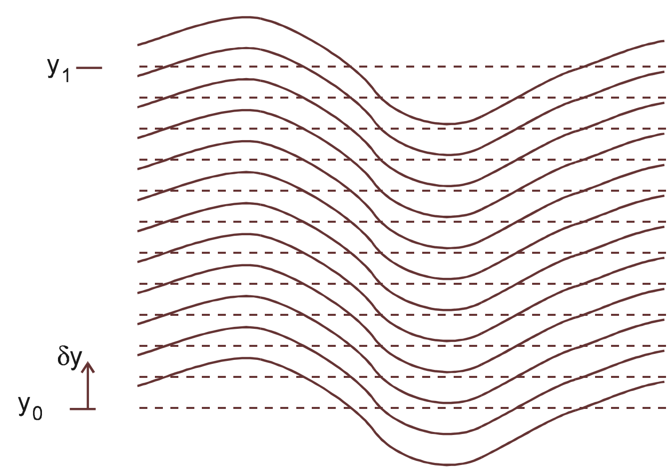
$$\rightarrow \tan\gamma \approx \gamma \approx \partial(\delta z_g)/\delta x \approx -\frac{\partial\theta}{\partial x}/\frac{\partial\theta}{\partial z}$$

so $\delta z_g \approx -\theta'/(\partial\bar{\theta}/\partial z)$.
C is the surface of *constant geometric height* z_g reference position for $\mathcal{C}$.
p' the pressure variation along C, then, along $\mathcal{C}$, $\delta p = p' - g\rho\delta z_g$. So

$$\overline{\gamma\ \delta p} = \overline{\frac{\partial(\delta z_g)}{\partial x}p'} - g\rho\overline{\frac{\partial(\delta z_g)}{\partial x}\delta z_g} = \overline{\frac{\partial(\delta z_g)}{\partial x}p'}$$

$$= -\overline{\delta z_g\frac{\partial p'}{\partial x}} = f_0\rho\frac{\overline{v'\theta'}}{\partial\bar{\theta}/\partial z} ,$$

$$\rightarrow \quad \tau = -f_0\frac{\overline{v'\theta'}}{\partial\bar{\theta}/\partial z}$$



→ so $F^{(z)}$ represents vertical momentum transport by *form drag* on isentropic surfaces.
So (unlike *e.g.*, chemical tracers) momentum can be *radiated* over large distances.

(vi) Rossby waves

- Barotropic
- Baroclinic
- Rossby wave breaking

Barotropic Rossby waves

Two-dimensional flow ($\partial/\partial z = 0$)

PV is just absolute vorticity $q = f_0 + \beta y + \nabla_h^2 \psi$
($\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$)

Vorticity conservation for waves on a constant zonal flow $\bar{u}$,
$\rightarrow \partial\bar{q}/\partial y = \beta$

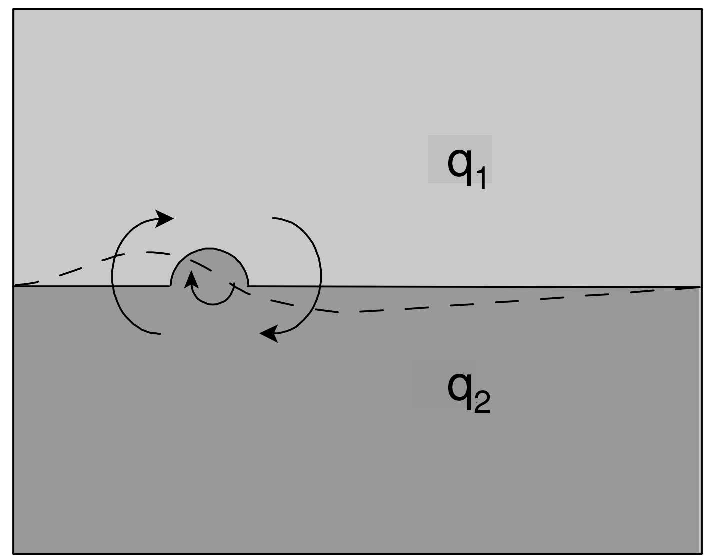
$$\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x} + \beta v' = 0$$

$$\rightarrow \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla_h^2\psi' + \beta\frac{\partial \psi'}{\partial x} = 0$$

wave solutions $\psi' = \mathrm{Re}[\Psi_0 \exp\{i(kx + ly - kct)\}]$ where

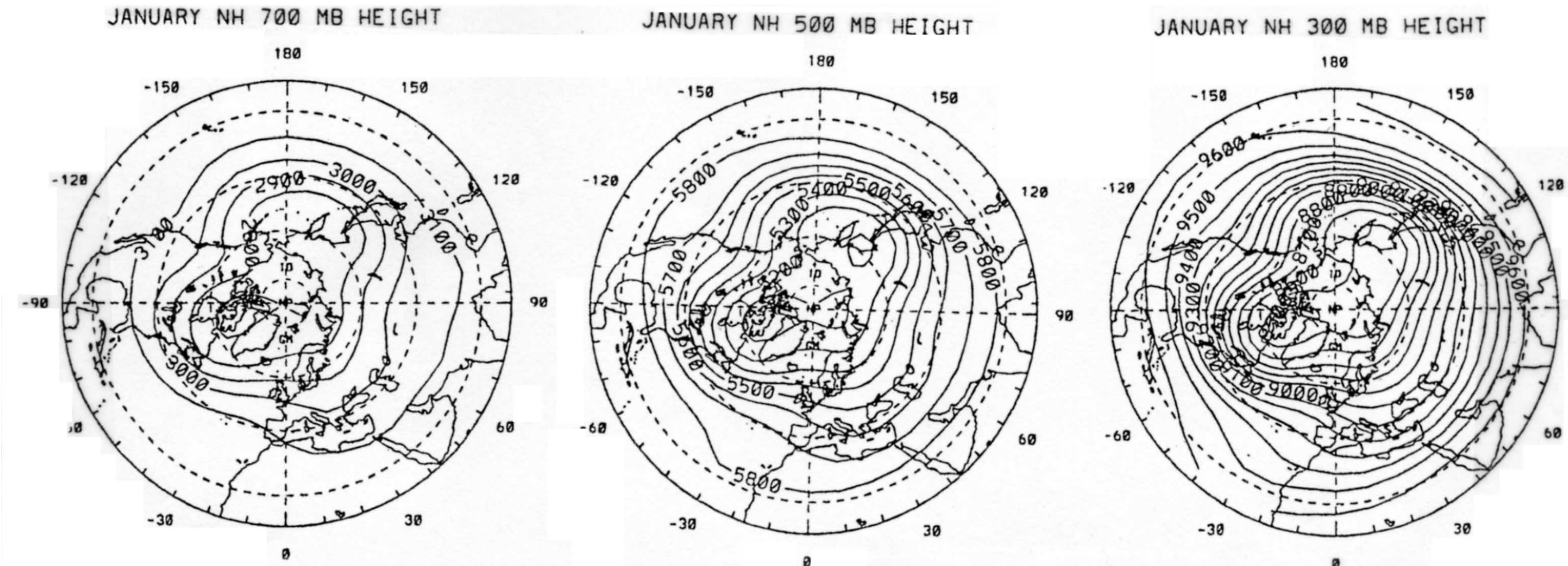
$$c = \bar{u} - \frac{\beta}{k^2 + l^2}$$

"elasticity" of PV gradient
→ westward propagation (relative to mean flow)
→ dispersive



Stationary Rossby waves

Long-term January mean geopotential height



NCEP data

Barotropic stationary waves:

$$c = \bar{u} - \frac{\beta}{k^2 + l^2} \quad \rightarrow \quad \kappa_s^2 = k^2 + l^2 = \frac{\beta}{\bar{u}}$$

For $\beta = 1.6 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$, $\bar{u} = 30\mathrm{ms}^{-1}$,

$$\frac{2\pi}{\kappa_s} = 2\pi\sqrt{\frac{\bar{u}}{\beta}} \simeq 8600 \text{ km}$$

$$\simeq \text{ zonal wave 3 at } 45^o \text{ latitude}$$

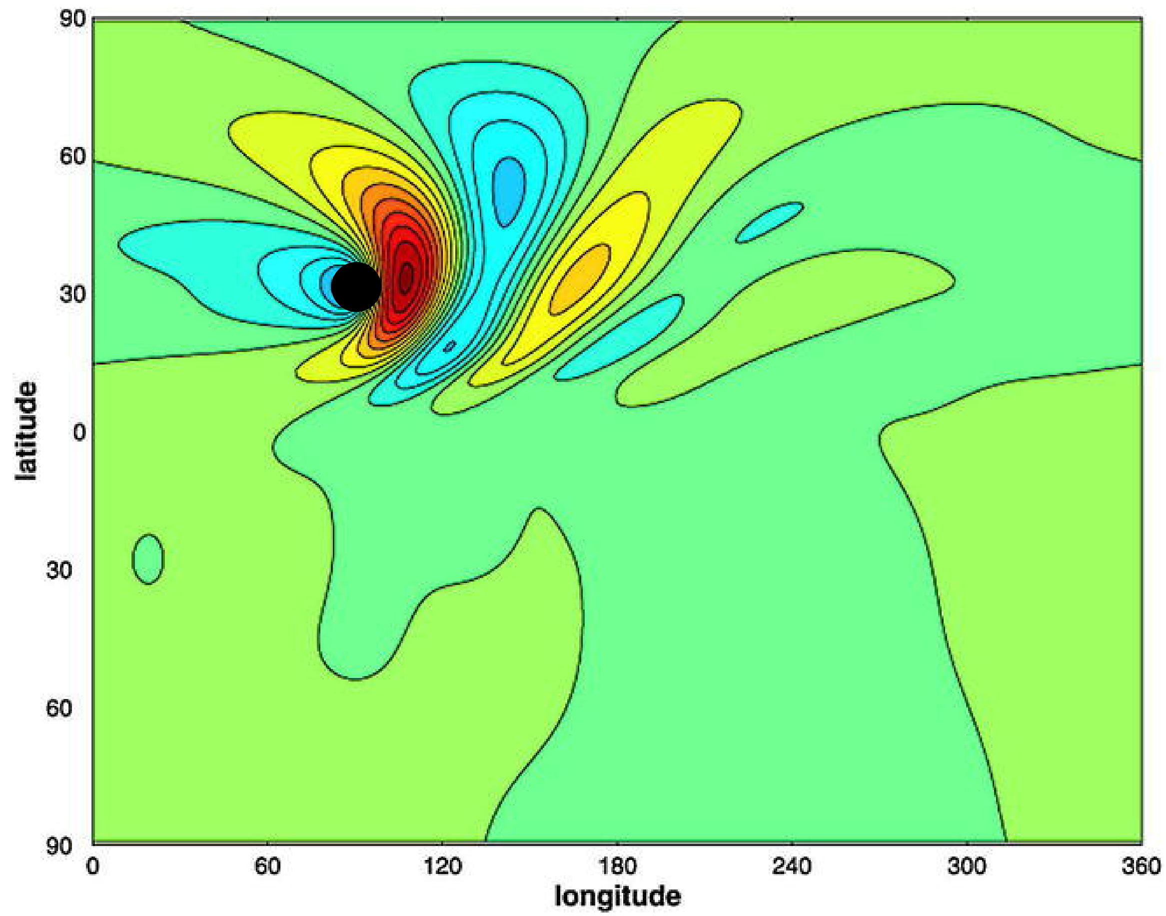
Zonal group velocity of stationary waves:

$$c_{g,x} = \frac{\partial(ck)}{\partial k} = \bar{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}$$

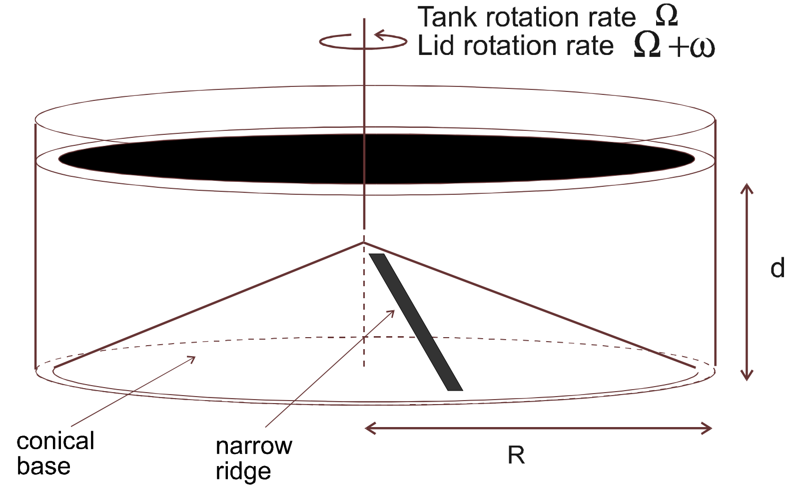
$$= 2k^2\frac{\bar{u}^2}{\beta} > 0$$

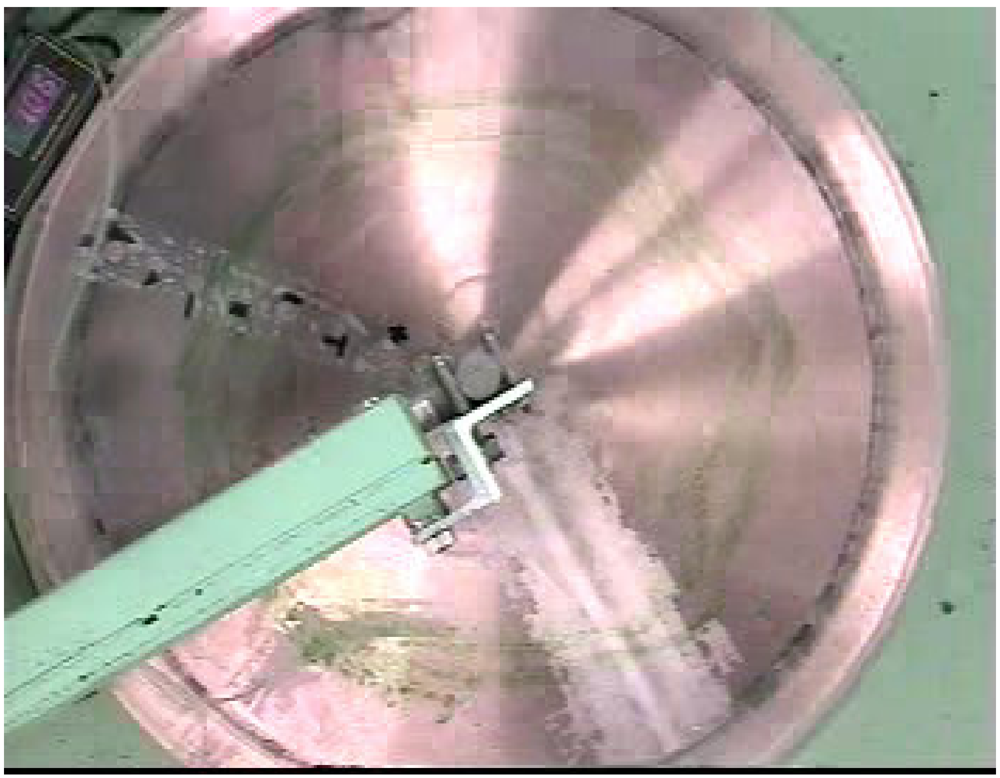
Rossby wave propagation on the sphere from a localized midlatitude source [Held 1983]

Realistic zonal winds (with tropical easterlies)



Stationary Rossby waves in the lab



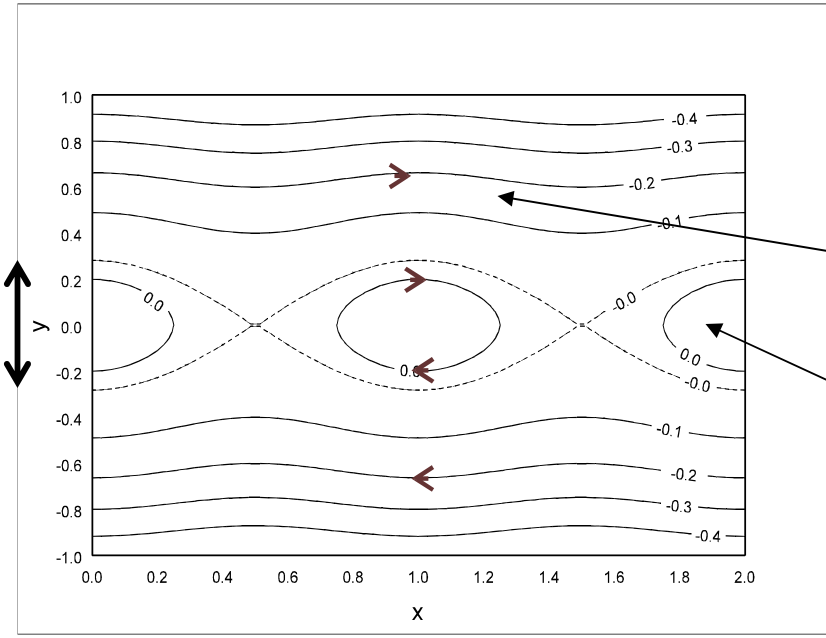


Critical layers and Rossby wave breaking

$$\bar{u} \simeq \Lambda y$$

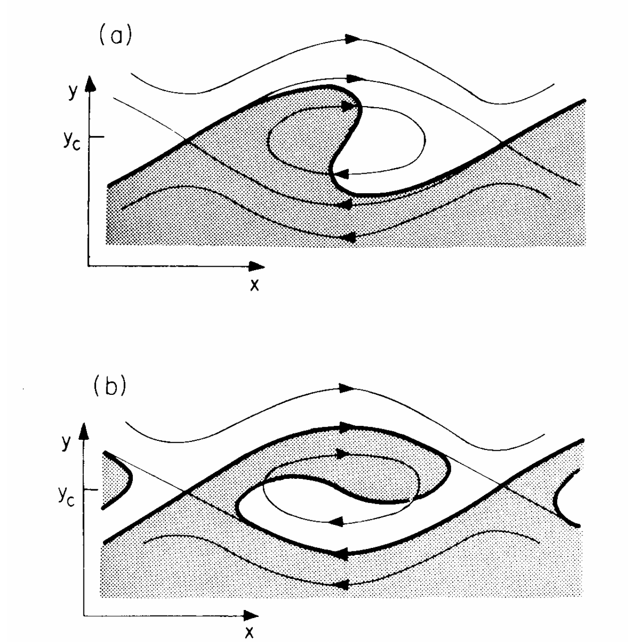
$$\psi = -\frac{1}{2}\Lambda y^2 + \Psi(0)\cos kx$$

$$\text{width} = 4\sqrt{\frac{\Psi(0)}{\Lambda}}$$



Mean westerlies: wavy streamlines

Closed eddies: overturning:



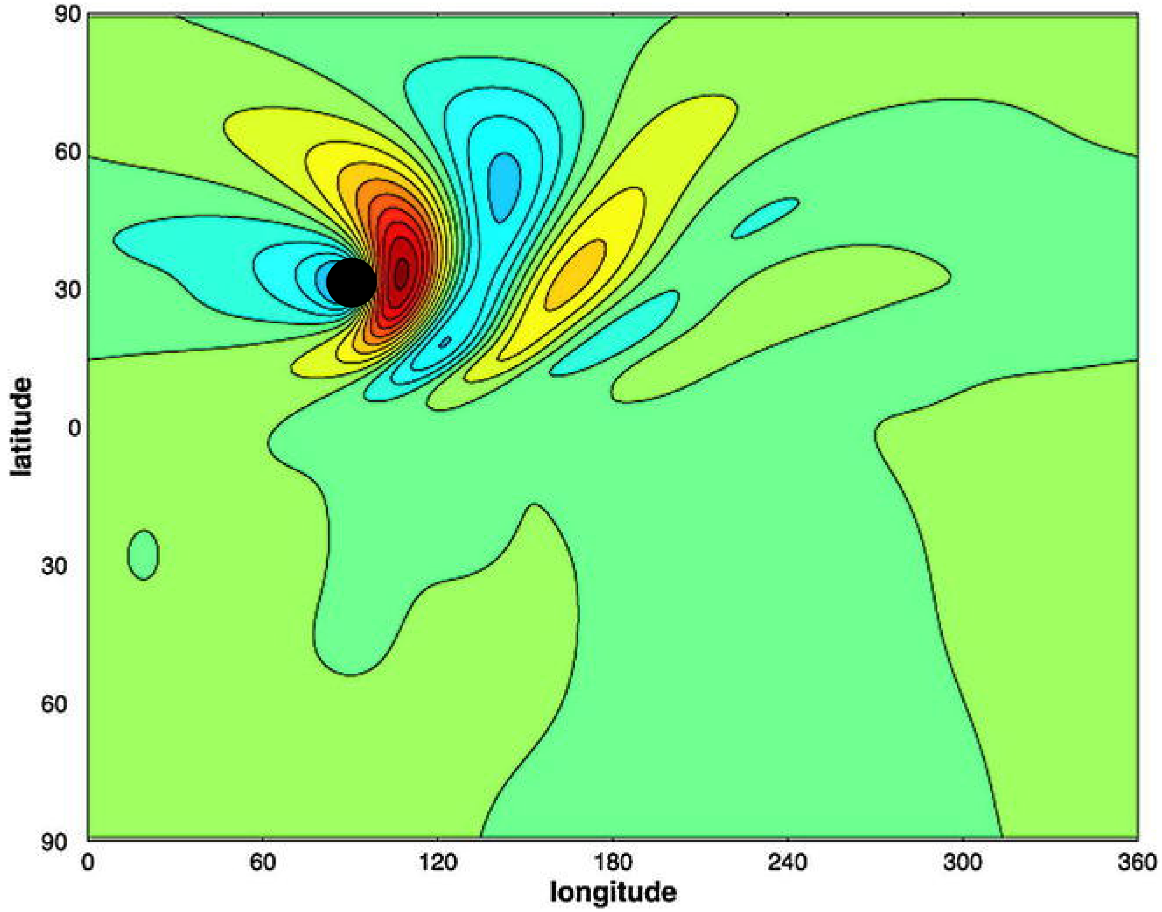
Haynes (1985)

$$\overline{\frac{\partial q}{\partial y}} > 0 \quad \rightarrow \quad \overline{v'q'} < 0 \quad \rightarrow \quad \nabla \cdot \mathbf{F} < 0$$

- absorption of wave activity

Rossby wave propagation on the sphere from a localized midlatitude source [Held 1983]

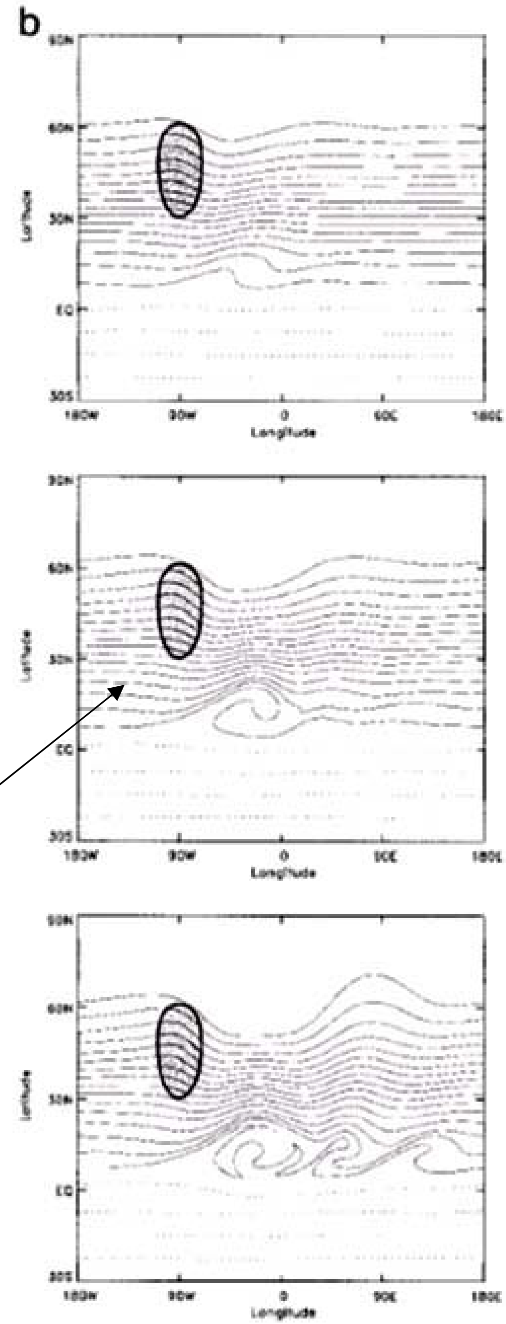
Realistic zonal winds (with tropical easterlies)



Subtropical breaking of Rossby waves from a localized midlatitude source

(1-layer; 300 hPa mean wind)

[Esler et al., *J Atmos Sci*, 2000]

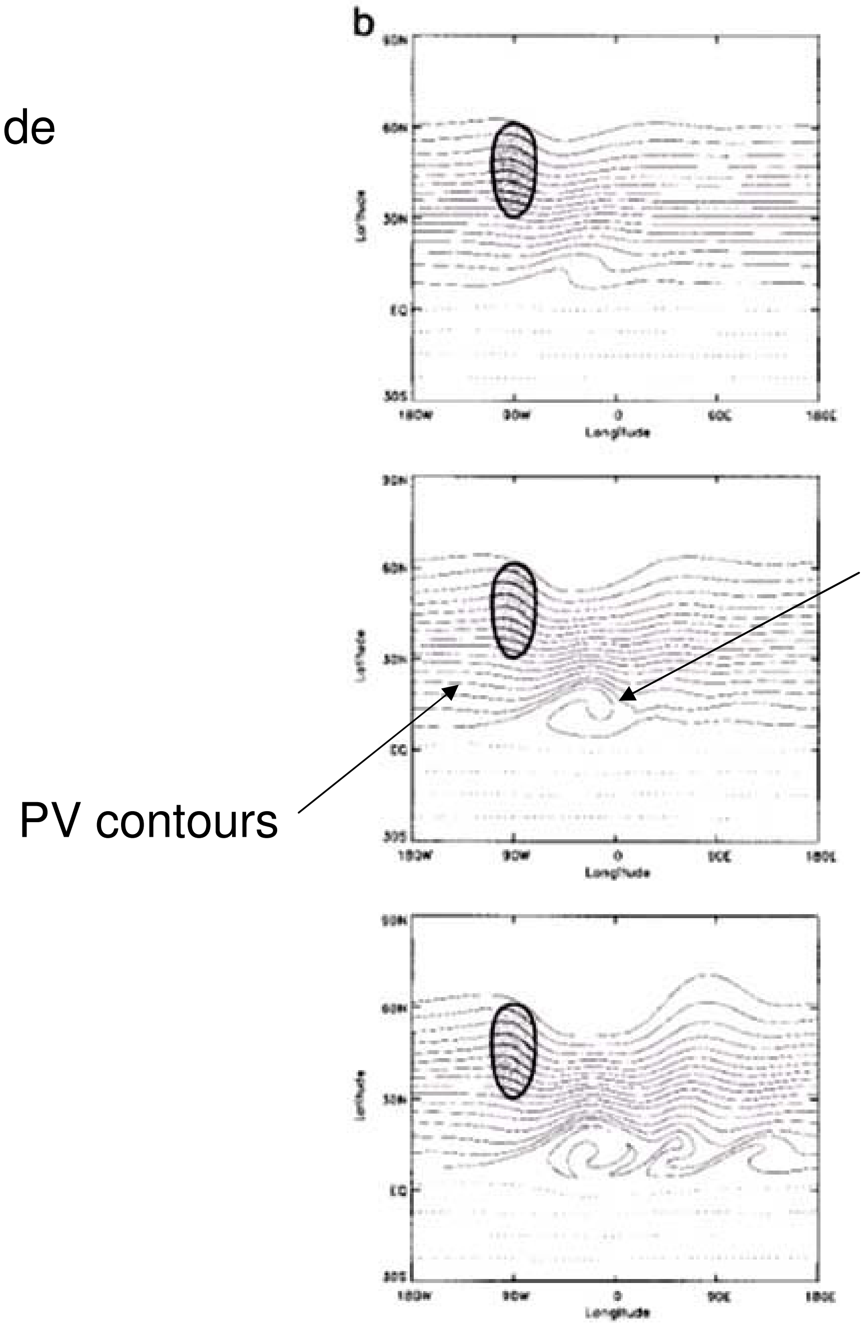


PV contours

Subtropical breaking of Rossby waves from a localized midlatitude source

(1-layer; 300 hPa mean wind)

[Esler et al., *J Atmos Sci*, 2000]



Baroclinic Rossby waves: Vertical propagation
[Charney & Drazin, *J. Geophys. Res.*, **66**, p83, 1961]

Conservative, small amplitude waves on constant background flow $\bar{u}$, N^2 also constant
Linearized QGPV equation:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + v'\frac{\partial \bar{q}}{\partial y} = 0$$

where now

$$q' = \Delta^2\psi' \equiv \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{\rho f_0^2}{N^2}\frac{\partial\psi'}{\partial z}\right),$$

$$\frac{\partial \bar{q}}{\partial y} = \beta$$

so

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\Delta^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0$$

Recall $\rho = \rho_0\exp(-z/H)$. Solutions are of the form

$$\psi' = \mathrm{Re}\,\Psi_0\ \exp\left(\frac{z}{2H}\right)\ \exp[i(kx + ly + mz - kct)]$$

where

$$\boxed{m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{(\bar{u} - c)} - k^2 - l^2\right) - \frac{1}{4H^2}}$$

or

$$c - \bar{u} = -\beta\left(k^2 + l^2 + \frac{f_0^2}{N^2}m^2 + \frac{f_0^2}{4N^2H^2}\right)^{-1}$$

→ dispersion relation for baroclinic Rossby waves

Vertical propagation of stationary waves

Vertical wavenumber m for $c = 0$

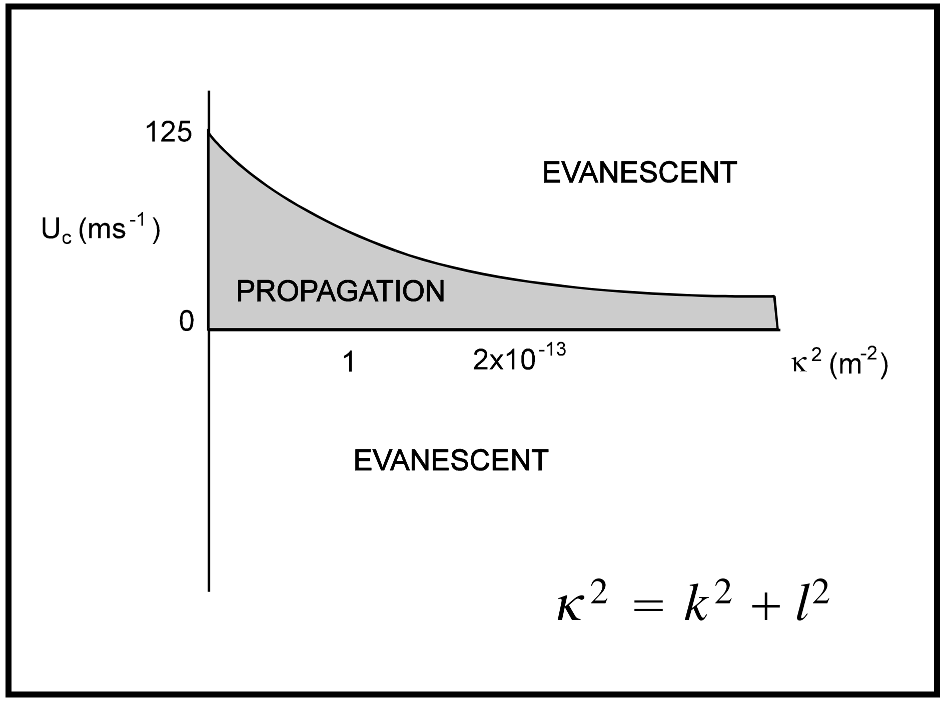
$$m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{\bar{u}} - k^2 - l^2\right) - \frac{1}{4H^2}$$

real m requires

$$0 < \bar{u} < U_c$$

"Rossby critical velocity" U_c is

$$U_c = \beta\left(k^2 + l^2 + \frac{f_0^2}{4N^2H^2}\right)^{-1}$$



→ propagation "window" for the mean winds

→ no propagation through easterlies $\bar{u} < 0$, nor strong westerlies $\bar{u} > U_c$

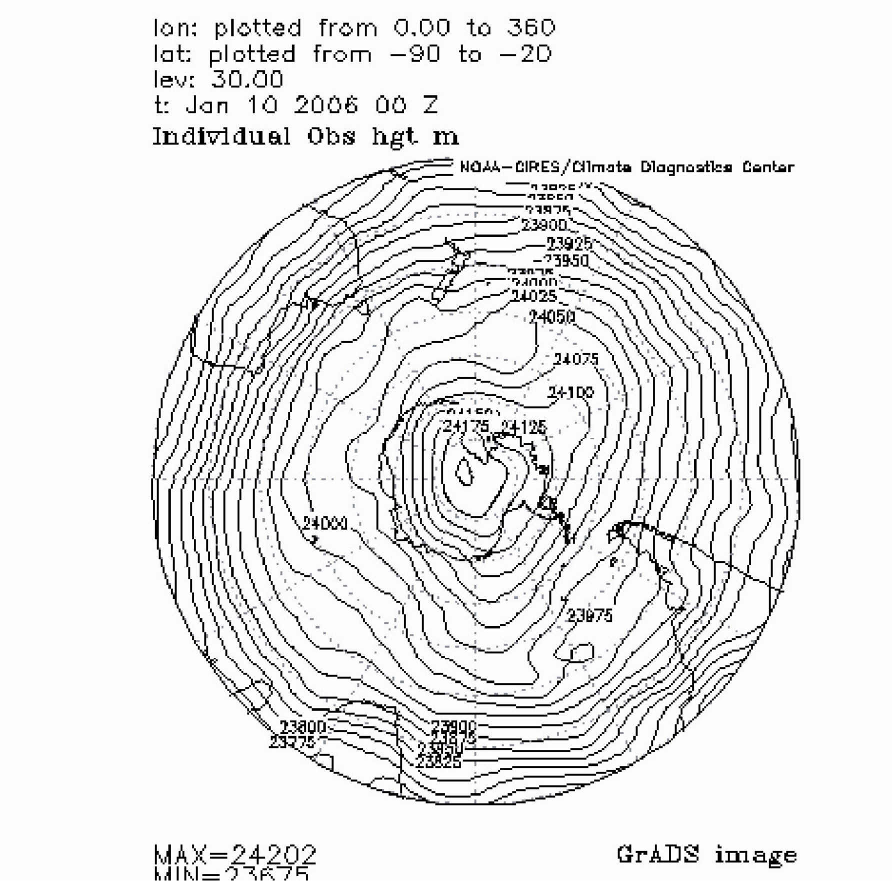
U_c decreases with increasing $k^2 + l^2$, so the window becomes narrow for small-scale waves

synoptic scale wave, $\kappa^2 = 1.96\ 10^{-11} m^{-2}$, $U_c \simeq 1 ms^{-1}$

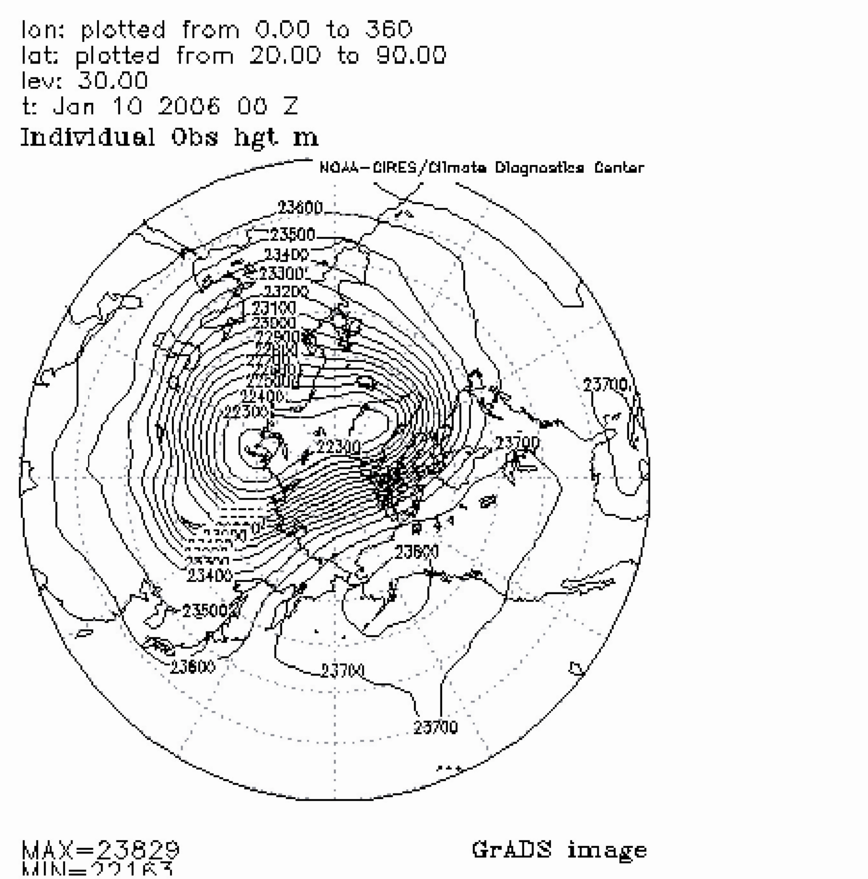
largest planetary scale wave $k = \pi/(14000km)$, $l = \pi/(6000km)$, $U_c \simeq 35 ms^{-1}$

Typical stratospheric analyses (30hPa, 2006 Jan 10)

summer



winter



almost no waves

planetary scales only

NCEP data

References

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- Charney, J. G., Stern, M. E., 1962: On the Stability of Internal Baroclinic Jets in a Rotating Atmosphere, J. Atmos. Sci., 19, 159-172,
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