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Lecture 2: Hamiltonian methods

- Hamiltonian formulation of the QG equations
  - Hamiltonian
  - Poisson bracket
  - Casimir invariants
  - Conserved properties
- Stability
  - Instabilities
- Contour dynamics
- Equilibria – simulated annealing and Dirac constraints
  - Dirac bracket

(with Phil Morrison, UT Austin)

1 —Hamiltonian formulation of the QG equations

We shall derive this for the case of the 1.5 layer QG equations with the possibility of steady flows in the deep layer (or topography); extension to the 2 layer model is straightforward.

$$\frac{\partial}{\partial t} q = [q, \psi] \quad , \quad q = \nabla^2\psi - \gamma^2\psi + \beta y + \gamma^2\psi_2 \equiv \nabla^2\psi - \gamma^2\psi + T$$

with

$$T(\mathbf{x}) = \beta y + \gamma^2\psi_2$$

We will generally just assume $T(\mathbf{x})$ is steady, but specialize to the case $T(y)$ or $T = 0$ at times. Our equations become

$$\begin{aligned} \psi(\mathbf{x}) &= \int d\mathbf{x}'\, G(\mathbf{x}-\mathbf{x}')\Big(q(\mathbf{x}') - T(\mathbf{x}')\Big) \\ \frac{\partial}{\partial t} q &= [\, q, \psi \,] \end{aligned} \tag{QG1}$$

1.1 — Hamiltonian

To relate the Hamiltonian to q, we start with the upper layer flow energy

$$H = \frac{1}{2}\int d\mathbf{x}\, |\nabla\psi|^2 + \gamma^2\psi^2$$

(with the first term being the kinetic energy and the second the available potential energy) and show that it is conserved

$$\frac{\partial}{\partial t}H = \int \nabla\psi\cdot\nabla\frac{\partial\psi}{\partial t} + \gamma^2\psi\frac{\partial\psi}{\partial t} = \int -\psi\frac{\partial}{\partial t}(\nabla^2\psi - \gamma^2\psi) = \int -\psi\frac{\partial}{\partial t}q = \int \psi[\psi, q] = 0$$

Note that conservation of upper layer energy is possible here becuase T is not time-dependent. In the full two-layer context, you can exchange energy between the layers. The Hamiltonian is

$$H[q] = -\frac{1}{2}\int\psi(q-T) = -\frac{1}{2}\iint d\mathbf{x}\, d\mathbf{x}' \Big(q(\mathbf{x}) - T(\mathbf{x})\Big)G(\mathbf{x}-\mathbf{x}')\Big(q(\mathbf{x}') - T(\mathbf{x}')\Big) \qquad (H)$$

where it has been expressed as a *functional* of q (a map from a function to a real number). To continue, we will need the functional (or variational) derivative of H. For

$$F[q] = \int d\mathbf{x}\, f(q(\mathbf{x}))$$

with f an ordinary function taking the real number $q(\mathbf{x})$ into another real number. The functional derivative of $F[q]$ is

$$\begin{aligned} F[q+\delta q] - F[q] &\equiv \int \frac{\delta F}{\delta q}\delta q \\ &= \int f(q(\mathbf{x}) + \delta q(\mathbf{x})) - f(q(x)) \\ &= \int f'(q(\mathbf{x}))\delta q(\mathbf{x}) \\ \Rightarrow \frac{\delta F}{\delta q} &= f'(q(\mathbf{x})) \end{aligned}$$

and is a function of $\mathbf{x}$.[†] The functional derivative of $H[q]$ is

$$\begin{aligned} \delta H &= -\frac{1}{2}\iint \delta q(\mathbf{x})G(\mathbf{x}-\mathbf{x}')\Big(q(\mathbf{x}') - T(\mathbf{x}')\Big) - \frac{1}{2}\iint \Big(q(\mathbf{x}) - T(\mathbf{x})\Big)G(\mathbf{x}-\mathbf{x}')\delta q(\mathbf{x}') \\ &= \int \delta q(\mathbf{x}) \int -G(\mathbf{x}-\mathbf{x}')\Big(q(\mathbf{x}') - T(\mathbf{x}')\Big) \end{aligned}$$

giving

$$\frac{\delta H}{\delta q} = -\psi \qquad (1.1)$$

[†] In more general cases, f will be an operator and $\frac{\delta F}{\delta q}$ will be related to the adjoint operator.

1.2 — Poisson bracket

The Poisson bracket is analogous to the $[A, B]$ operator but operating on functional rather than functions. It makes the Hamiltonian formulation into a Lie algebra: it is a bilinear, antisymmetric operator satisfying the triple product (Jacobi) identity

$$\{A, B + \lambda C\} = \{A, B\} + \lambda\{A, C\} \quad , \quad \{A, B\} = -\{B, A\} \quad ,$$

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$$

(Note that these hold for the vector cross product and for $[A, B]$ which is related to the cross product of ∇A and ∇B.)

If we make the point vortex approximation and neglect T

$$q = \sum q_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

we have

$$H = -\frac{1}{2}{\sum}'{\sum}' q_i q_j G(\mathbf{x}_i - \mathbf{x}_j) \quad , \quad \psi(\mathbf{x}) = \sum q_j G(\mathbf{x} - \mathbf{x}_j)$$

The dynamical equations

$$\frac{\partial}{\partial t} x_i = -\frac{\partial \psi}{\partial y_i} = \frac{1}{q_i}\frac{\partial H}{\partial y_i}$$
$$\frac{\partial}{\partial t} y_i = \frac{\partial \psi}{\partial x_i} = -\frac{1}{q_i}\frac{\partial H}{\partial x_i}$$

imply

$$\frac{\partial}{\partial t} f(\mathbf{x}_1, \mathbf{x}_2, ...) = \sum \frac{1}{q_i}\left(\frac{\partial f}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial H}{\partial x_i}\right)$$

giving

$$\{A, B\} = \frac{1}{q_i}\left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial y_i} - \frac{\partial A}{\partial y_i}\frac{\partial B}{\partial x_i}\right)$$

This is the Hamiltonian form of the point vortex model.

To transition to the continuum, we want to relate functionals of q to functions of the coordinates of the Lagrangian points: let

$$F[q] = f(\mathbf{x}_i)$$

so that

$$\delta f = \delta \mathbf{x}_i \cdot \nabla f$$

and

$$\begin{aligned}\delta F &= \int \frac{\delta F}{\delta q} q_i[-\delta \mathbf{x}_i \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_i)] \\ &= q_i \int \delta(\mathbf{x} - \mathbf{x}_i)\delta \mathbf{x}_i \cdot \nabla\left(\frac{\delta F}{\delta q}\right) \\ &= q_i \delta \mathbf{x}_i \cdot \nabla\left(\frac{\delta F}{\delta q}\right)\Big|_{\mathbf{x}_i}\end{aligned}$$

giving

$$\frac{\partial f}{\partial x_i} = q_i \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q}\right)\Big|_{\mathbf{x}_i}$$

The Poisson bracket becomes

$$\begin{aligned}\{A, B\} &= q_i \frac{\partial}{\partial x}\left(\frac{\delta A}{\delta q}\right)\frac{\partial}{\partial y}\left(\frac{\delta B}{\delta q}\right) - q_i \frac{\partial}{\partial y}\left(\frac{\delta A}{\delta q}\right)\frac{\partial}{\partial x}\left(\frac{\delta B}{\delta q}\right)\\ &= \int d\mathbf{x}\, q\left[\frac{\delta A}{\delta q}, \frac{\delta B}{\delta q}\right]\end{aligned}$$

If we now put T back in, we have

$$\begin{aligned}\{A, B\} &= \int d\mathbf{x}\; q\left[\frac{\delta A}{\delta q}, \frac{\delta B}{\delta q}\right]\\ H &= -\frac{1}{2}\iint d\mathbf{x}\, d\mathbf{x}'\,\Big(q(\mathbf{x}) - T(\mathbf{x})\Big)G(\mathbf{x}-\mathbf{x}')\Big(q(\mathbf{x}') - T(\mathbf{x})\Big)\end{aligned} \qquad (H-QG)$$

To verify that this is correct, note that

$$\frac{\delta H}{\delta q} = -\psi$$

so we need to check that

$$\frac{\partial}{\partial t}F[q] = \int d\mathbf{x}\, q\left[\frac{\delta F}{\delta q}, -\psi\right] = \int d\mathbf{x}\,\frac{\delta F}{\delta q}[q, \psi]$$

consistent with

$$\frac{\partial}{\partial t}F[q] = \int d\mathbf{x}\,\frac{\delta F}{\delta q}\frac{\partial}{\partial t}q \quad \text{and} \quad \frac{\partial}{\partial t}q = [q, \psi]$$

Alternatively, we can note that

$$q(\mathbf{x}) = \int d\mathbf{x}' q(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')$$

so that $q(\mathbf{x})$ is a functional of q with

$$\frac{\delta q(\mathbf{x}')}{\delta q(\mathbf{x})} = \delta(\mathbf{x}'-\mathbf{x})$$

and

$$\begin{aligned}\frac{\partial}{\partial t}q(\mathbf{x}') &= \{q(\mathbf{x}'), H\}\\ &= \int d\mathbf{x}\, q(\mathbf{x})[\delta(\mathbf{x}'-\mathbf{x}), -\psi(\mathbf{x})]\\ &= \int d\mathbf{x}\, \delta(\mathbf{x}'-\mathbf{x})[q(\mathbf{x}), \psi(\mathbf{x})]\\ &= [q(\mathbf{x}'), \psi(\mathbf{x}')]\end{aligned}$$

as required.

1.3 — Casimir invariants

The point vortex equations are in canonical form; this is possible for even order ODE sets, but not for odd orders. I.e., if we write

$$\{A, B\} = \frac{\partial A}{\partial z_i} J_{ij} \frac{\partial B}{\partial z_j}$$

J will be an antisymmetric matrix; an odd order matrix will have at least one 0 eigenvalue. So there exists a $\mathcal{C}(z)$ such that $\nabla_z \mathcal{C}$ is an eigenvector in the null space of **J** and

$$\{\mathcal{C}, B\} = 0$$

for all B. Similarly, in our continuum form, any functional of the form

$$\mathcal{C}[q] = \int d\mathbf{x}\, C(q)$$

(where it is important that C has no explicit dependence on $\mathbf{x}$) will satisfy $\{\mathcal{C}, B\} = 0$ for any B. These are called "Casimir invariants" or "Casimirs." To demonstrate that $\mathcal{C}[q]$ has that property

$$\{\mathcal{C}, B\} = \int q \left[C'(q), \frac{\delta B}{\delta q} \right] = \int \frac{\delta B}{\delta q} [\, q, C'(q)\,] = \int \frac{\delta B}{\delta q} [\, q, q\,]\, C''(q) = 0$$

This is equivalent to the conservation of the integral of any power of q – e.g., the net PV or potential enstrophy, etc. But, more broadly, we conserve the PDF of q or the area between q contours. From the point vortex viewpoint, this restrict the dynamics such that $\frac{\partial}{\partial t} q_i = 0$ – Lagrangian conservation of q.

1.4 — Conserved properties

Unlike the Casimirs, conserved properties such as linear or angular momentum depend on symmetries of the Hamiltonian. Noether's theorem makes that explicit: suppose we have a continuous symmetry $\mathbf{x} \to \mathbf{x} + \mathbf{e}(\mathbf{x})$ meaning

$$H[q(\mathbf{x} + \mathbf{e})] = H[q(\mathbf{x})]$$

then, for small $|\mathbf{e}|$,

$$H[q + \mathbf{e} \cdot \nabla q] = H[q] \quad \text{or} \quad \int d\mathbf{x} \frac{\delta H}{\delta q} \mathbf{e} \cdot \nabla q = 0$$

If we let $\mathbf{e} = (\phi_y, -\phi_x)$ so that the transformation is area-preserving,

$$\int d\mathbf{x} \frac{\delta H}{\delta q} [q, \phi] = \int d\mathbf{x}\, q \left[\phi, \frac{\delta H}{\delta q} \right] = 0$$

If we now define
$$L[q] = \int d\mathbf{x}\phi(\mathbf{x})q(\mathbf{x}) \quad \Rightarrow \quad \phi = \frac{\delta L}{\delta q}$$
then $\frac{\partial}{\partial t}L = 0$.

The symmetries we will use are

- zonal translation $\mathbf{e} = (1, 0)$, $\phi = y$, $P = \int yq$, corresponding to zonal momentum in the barotropic case $\int u$.
- if $T = 0$, we also have meridional translation $(-\int xq)$ and angular momentum $\mathbf{e} = (-y, x)$, $\phi = -\frac{1}{2}(x^2 + y^2)$ with $L = -\frac{1}{2}\int |\mathbf{x}|^2 q$
- conservation of H is related to time-translation symmetry

2 —Stability

The Liaponov form of stability asks whether two states which are initially close together remain close together; i.e. if
$$\frac{\partial}{\partial t}\overline{q} = \{\overline{q}, H[\overline{q}]\} \qquad (basic\ state)$$
and
$$\frac{\partial}{\partial t}\overline{q} + \frac{\partial}{\partial t}q' = \{\overline{q} + q', H[\overline{q} + q']\}$$
or
$$\frac{\partial}{\partial t}q' = \{\overline{q} + q', H[\overline{q} + q']\} - \{\overline{q}, H[\overline{q}]\} \qquad (pert)$$
and $||q'(0)|| << ||\overline{q}(0)||$ does $||q'(t)||$ remain small? We take a norm which is conserved such as H or $\int q^2$ so that a measure of the perturbation amplitude is $(||q'(t)||/||\overline{q}||)^{1/2}$ and we can talk about whether this can be bounded by some small number ϵ for all initial perturbations with amplitude $< \delta$ but without restrictions on the shape.

The perturbation equation can also be written as
$$\begin{aligned}\frac{\partial}{\partial t}q' &= [\overline{q} + q', \int G(\mathbf{x} - \mathbf{x}')(\overline{q}(\mathbf{x}') + q'(\mathbf{x}'))] - [\overline{q}, \int G(\mathbf{x} - \mathbf{x}')\overline{q}(\mathbf{x}')] \\ &= [\overline{q} + q', \int G(\mathbf{x} - \mathbf{x}')q'(\mathbf{x}')] + [q', \int G(\mathbf{x} - \mathbf{x}')\overline{q}(\mathbf{x}')] \\ &= [\overline{q} + q', \psi'] + [q', \overline{\psi}]\end{aligned}$$

cos E cos2 E coscos E coscos2 E

The linearized problem
$$\frac{\partial}{\partial t}q' = [\overline{q}, \psi'] + [q', \overline{\psi}] \qquad (lin\ pert)$$
can give insight into whether or not a given flow is unstable and what the growth rates could be.

Most of the work on stability theory as applied to GFD has been for steady flows and uses the linearized equations *(lin pert)*. We will try to remain fairly general.

We can prove stability (not just linearized) if we can find a functional $A[q]$ which is

- conserved
- has $\frac{\delta A}{\delta q} = 0$ at $\overline{q}$
- is either positive definite or negative definite in some region surrounding $\overline{q}$ i.e., for all q' with $||q'(0)||/||\overline{q}|| < \delta$. Indeed, we can use $|A|$ itself as a norm, though we might choose to use a different form.

Although H is conserved, it will not generally have zero gradient at the basic state. But, as Arnold recognized, the Casimirs give more flexibility. Let us define

$$A[q] = H[q] + \lambda L[q] + \mathcal{C}[q]$$

where $L = \int \phi q$ is one of the conserved properties (if such exist) and $\mathcal{C}$ is a yet-unspecified Casimir $\mathcal{C} = \int C(q)$. The Lagrange multiplier, λ is also a free parameter.

The first variation is

$$\frac{\delta A}{\delta q} = -\psi + \lambda\phi + C'(q)$$

For general time-dependent states, it will not be possible to make this zero – even though A is constant, the gradient can change with time. An analogy would be particle motion in a fixed flow with a streamfunction Ψ: even though the value of Ψ is conserved, the distance between streamlines is not necessarily fixed. For a steady flow, however, $\psi = \Psi(q)$, so we can find a $C(q)$ which will make this zero. In particular, if q, $\overline{\psi}$ and ϕ depend on only one coordinate, χ, we can invert to find $\chi(q)$ and write

$$C(q) = \int^q dq' \left(\overline{\psi}(q') - \lambda\phi(q')\right)$$

- zonal flow: here we can select $\phi = y$ (conservation of a zonal momentum-like variable)

$$C(q) = \int^q dq' \left(\overline{\psi}(q') - \lambda y(q')\right)$$

  This can also work if $T = T(y)$.
- circular vortices: $\phi = -\frac{1}{2}r^2$ (angular momentum),

$$C(q) = \int^q dq' \left(\overline{\psi}(q') + \frac{1}{2}\lambda r^2(q')\right)$$

- steady flows (non-symmetric) such as the Fofonoff gyre have

$$C(q) = \int^q dq' \, \overline{\psi}(q')$$

  with the gyre case having $\overline{\psi} = \alpha q$.

- steadily propagating disturbances $\overline{\psi}(x-ct,y)$: if we choose $\lambda = c$ and $\phi = -y$, then

$$C'\big(q(x-ct,y)\big) = \psi(x-ct,y) + cy$$

which will again have a solution.

All of these have $C = \int^q dq'\,\Psi(q')$ with various forms of Ψ. Using this in the conserved Arnold invariant yields

$$\begin{aligned}
A[\overline{q}+q'] - A[\overline{q}] &= H[\overline{q}+q'] - H[\overline{q}] + \int \delta\mathbf{x}\,\phi q' + \int d\mathbf{x}\,\big(C(\overline{q}+q') - C(\overline{q})\big) \\
&= -\iint q'G(\mathbf{x}-\mathbf{x}')(\overline{q}-T) - \frac{1}{2}\iint q'Gq' + \int d\mathbf{x}\,\phi q' + \int d\mathbf{x}\,\big(C(\overline{q}+q') - C(\overline{q})\big) \\
&= -\frac{1}{2}\iint q'Gq' - \int d\mathbf{x}\,q'C'(\overline{q}) + \int d\mathbf{x}\,\big(C(\overline{q}+q') - C(\overline{q})\big)
\end{aligned}$$

(using the vanishing of $\frac{\delta A}{\delta q}$ at $q = \overline{q}$. The first term is positive definite, so the flow will be stable if

$$\int d\mathbf{x}\Big(C(\overline{q}+q') - C(\overline{q}) - C'(\overline{q})q'\Big) > 0$$

or is sufficiently negative to overcome the positive term. The former will hold if

$$\int^{\overline{q}+q'} ds\,\Psi(s) - \int^{\overline{q}} ds\,\Psi(s) - \Psi(\overline{q})q' > 0$$

or

$$\int_0^{q'} ds\,[\Psi(\overline{q}+s) - \Psi(\overline{q})] > 0$$

which will be true if

$$C''(\overline{q}) = \frac{d\Psi}{dq}\Big|_{\overline{q}} > 0 \qquad (A\ stab)$$

For vortices, we have

$$\frac{d\Psi}{dq} = \frac{\frac{\partial}{\partial r}\Psi}{\frac{\partial}{\partial r}\overline{q}} = \frac{\overline{v}(r) + \lambda r}{\frac{\partial}{\partial r}\overline{q}}$$

If $\frac{\partial}{\partial r}\overline{q} > 0$, we choose $\lambda > min(\overline{v}(r)/r)$, proving that the flow is stable. Likewise it holds for $\frac{\partial}{\partial r}\overline{q} < 0$; the criterion *(A stab)* yields the Raleigh stability theorem for circular vortices. If $\frac{\partial}{\partial r}\overline{q} = 0$ at a single radius, $r = r_0$, we can choose $\lambda = -\overline{v}(r_0)/r_0$ and see that the flow will be stable when $\overline{v}(r) - \overline{v}(r_0)$ is the same sign as $\frac{\partial}{\partial r}\overline{q}$. (Fjørtoft's theorem).

3 —Equilibria – simulated annealing and Dirac constraints

One way to pose the problem of finding equilibrium states is to say that we want to rearrange all the fluid parcels, each keeping its PV, until H (or some other conserved property) reaches a maximum or minimum (Carnevale, et al., 1989). That implies that $\frac{\delta H}{\delta q} = 0$ at $q = \overline{q}$ and, therefore, $\frac{\partial}{\partial t} q = 0$. We can represent rearrangements by advection by some flow field

$$\frac{\partial}{\partial t} q = [q, \Psi] \quad \text{or} \quad \frac{\partial}{\partial t} F = \{F, S\}$$

but are free to specify Ψ and S (which is now not H). With this evolution equation

$$\frac{\partial}{\partial t} H = -\int d\mathbf{x} \frac{\delta S}{\delta q} [q, \psi]$$

and we can make H evolve towards higher or lower values if

$$\frac{\delta S}{\delta q} = -\Psi = -\int d\mathbf{x}' \, K(\mathbf{x}|\mathbf{x}') \, [q(\mathbf{x}')\psi(\mathbf{x}')]$$

with K a sign-definite kernel. I.e.,

$$\begin{aligned} \frac{\partial}{\partial t} H &= \int d\mathbf{x} \, q \, [-\psi, -\Psi] \\ &= \int d\mathbf{x} \, \Psi \, [q, \psi] \\ &= \iint d\mathbf{x} \, d\mathbf{x}' \, K(\mathbf{x}|\mathbf{x}') \, [q(\mathbf{x}'), \psi(\mathbf{x}')] \, [q(\mathbf{x}), \psi(\mathbf{x})] \end{aligned}$$

is positive definite for positive K.

H

simulated annealing

3.1 — Dirac bracket

This procedure can work well for cases with structure in the T field, but does not arrive at steadily propagating or rotating states. To do that, we ask for a maximum H with additional constraints. For rotating states, we require that the angular momentum have a particular value as set by the initial state. Translating states can deal with the linear momentum. The procedure involves defining a Dirac bracket

$$\{A, B\}_D = \{A, B\} + \frac{\{A, C_1\}\{C_2, B\}}{\{C_1, C_2\}} + \frac{\{A, C_2\}\{C_1, B\}}{\{C_2, C_1\}}$$

where $C_1[q]$ and $C_2[q]$ are functionals which will be conserved if we replace the Poisson bracket by the Dirac bracket. Indeed they are Casimirs

$$\{C_j, B\}_D = 0$$

Therefore, if we use this bracket in the simulated annealing procedure, H will increase as fluid parcels are rearranged (retaining their PV), but C_j and all the moments of q will be conserved.

DB

DBSA

min H

References

[1] Vallis, G.K., Carnevale, G.F., and Young, W.R., 1989: Extremal energy properties and construction of stable solutions of the Euler equations, J. Fluid Mech., 207, 133-152. (p9 Carnevales, et al.,1989 に対応)