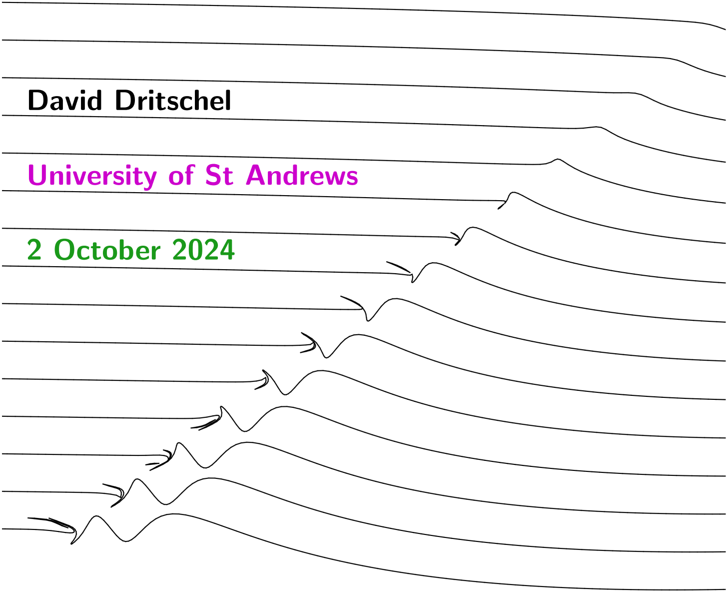
The onset of filamentation on two-dimensional vorticity interfaces



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Outline

Problem setting

We consider an ideal, inviscid, incompressible two-dimensional (2D) fluid governed by Euler's equation (1755):

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{\boldsymbol{\nabla} p}{\rho}$$

where $\boldsymbol{u} = (u, v)$ is the velocity field, p is pressure and ρ is (here) the constant density. We also have

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,.$$

Taking the 2D curl of Euler's equation results in

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0$$

where $\omega = \partial v/\partial x - \partial u/\partial y$ is the (scalar) vorticity (normal to the plane).

Problem setting

As the flow is 2D incompressible, we may satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ exactly by introducing a streamfunction ψ in terms of which

$$u = -\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial x} \,.$$

Then the definition of vorticity $\omega = \partial v/\partial x - \partial u/\partial y$ implies

$$\nabla^2 \psi = \omega \,,$$

a Poisson equation determining $\psi(\boldsymbol{x}, t)$ from the instantaneous distribution of $\omega(\boldsymbol{x}, t)$. *This is non-local.*

The evolution equation $\mathrm{D}\omega/\mathrm{D}t = 0$ is *nonlinear* because the advecting velocity field $\boldsymbol{u}$ depends on ω (non-locally as well).

Problem setting

Note that vorticity is materially-conserved, i.e. constant on *fluid particles.* This is a consequence of Kelvin's circulation theorem.

The infinite-dimensional dynamical system is Hamiltonian, with the kinetic energy serving as the Hamiltonian.

If the fluid domain has translational symmetry, then Kelvin's impulse

$$\boldsymbol{I} = \iint \omega\, \boldsymbol{x}\, \mathrm{d}x\mathrm{d}y$$

is also conserved.

If the fluid domain has rotational symmetry, then the angular impulse

$$J = \iint \omega\, (x^2 + y^2)\, \mathrm{d}x\mathrm{d}y$$

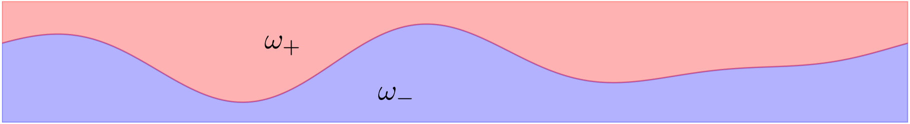
is also conserved.

Vorticity interfaces

Consider two fluid particles having the *same* vorticity, $\omega = \omega_0$, say.

If we exchange their positions, the distribution of $\omega(\boldsymbol{x}, t)$ is unaffected.
Therefore, this has no consequence for the flow evolution.

Now consider a *vorticity interface*, a curve $\mathcal{C}$ dividing the plane into two regions of *uniform* vorticity, ω_+ and ω_-.



The above 'particle exchange symmetry' means that only $\mathcal{C}$ and the jump in vorticity $\Delta\omega = \omega_+ - \omega_-$ across it matter in determining the velocity field $\boldsymbol{u}$.

Vorticity interfaces

Let $\mathcal{C}$ be directed such that vorticity ω_+ lies to its left, and ω_- lies to its right. ($\mathcal{C}$ can be open or closed.)

The formal solution of Poisson's equation $\nabla^2\psi = \omega$ in the entire plane is

$$
\begin{aligned}
\psi(\boldsymbol{x}, t) &= \frac{1}{2\pi} \iint \omega(\boldsymbol{x}', t) \log|\boldsymbol{x}' - \boldsymbol{x}| \,\mathrm{d}x\mathrm{d}y \\
&= \frac{\omega_+}{2\pi} \iint_{\mathcal{R}_+} \log|\boldsymbol{x}' - \boldsymbol{x}| \,\mathrm{d}x'\mathrm{d}y' + \\
&\quad \frac{\omega_-}{2\pi} \iint_{\mathcal{R}_-} \log|\boldsymbol{x}' - \boldsymbol{x}| \,\mathrm{d}x'\mathrm{d}y'
\end{aligned}
$$

where $\mathcal{R}_+$ and $\mathcal{R}_-$ are the regions where $\omega = \omega_+$ and ω_-, respectively.

Vorticity interfaces

Consider the associated velocity field, $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$:

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{\omega_+}{2\pi}\iint_{\mathcal{R}_+}\left(-\frac{\partial}{\partial y},\,\frac{\partial}{\partial x}\right)\log|\boldsymbol{x}'-\boldsymbol{x}|\,\mathrm{d}x'\mathrm{d}y' + \frac{\omega_-}{2\pi}\iint_{\mathcal{R}_-}\left(-\frac{\partial}{\partial y},\,\frac{\partial}{\partial x}\right)\log|\boldsymbol{x}'-\boldsymbol{x}|\,\mathrm{d}x'\mathrm{d}y'\,.$$

However, the function $\log|\boldsymbol{x}'-\boldsymbol{x}|$ is symmetric in $\boldsymbol{x}'$ and $\boldsymbol{x}$.
Hence, the above can equally-well be written

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{\omega_+}{2\pi}\iint_{\mathcal{R}_+}\left(\frac{\partial}{\partial y'},\,-\frac{\partial}{\partial x'}\right)\log|\boldsymbol{x}'-\boldsymbol{x}|\,\mathrm{d}x'\mathrm{d}y' + \frac{\omega_-}{2\pi}\iint_{\mathcal{R}_-}\left(\frac{\partial}{\partial y'},\,-\frac{\partial}{\partial x'}\right)\log|\boldsymbol{x}'-\boldsymbol{x}|\,\mathrm{d}x'\mathrm{d}y'\,.$$

Vorticity interfaces

Green's theorem (Stokes' theorem in the plane) tells us

$$\iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x'} - \frac{\partial P}{\partial y'} \right) = \int_{\mathcal{C}} P \, \mathrm{d}x' + Q \, \mathrm{d}y'$$

for (almost) any functions $P(x', y')$ and $Q(x', y')$. Here the contour $\mathcal{C}$ is traversed so that $\mathcal{R}$ is always on its left.

Therefore, taking $P = \log |\boldsymbol{x}' - \boldsymbol{x}|$ and $Q = 0$ for u, and taking $P = 0$ and $Q = \log |\boldsymbol{x}' - \boldsymbol{x}|$ for v, we have

$$\boldsymbol{u}(\boldsymbol{x}, t) = -\frac{\Delta\omega}{2\pi} \int_{\mathcal{C}} \log |\boldsymbol{x}' - \boldsymbol{x}| \, \mathrm{d}\boldsymbol{x}' \,,$$

a remarkably compact expression! The jump in vorticity $\Delta\omega$ arises because $\mathcal{C}$ is traversed in opposite directions in the two regions.

Vorticity interfaces

The *dynamics* cannot depend on fluid particles in the regions outside $\mathcal{C}$, since these particles can be exchanged arbitrarily with no effect on the velocity field $\boldsymbol{u}$.

$\Rightarrow$ Therefore, the dynamics is *entirely* dependent on $\mathcal{C}$.

We can deduce how $\mathcal{C}$ evolves by evaluating $\boldsymbol{u}$ on $\mathcal{C}$ and equating this to the material derivative of a particle *on* $\mathcal{C}$:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\frac{\Delta\omega}{2\pi}\int_{\mathcal{C}} \log|\boldsymbol{x}' - \boldsymbol{x}|\,\mathrm{d}\boldsymbol{x}'\,.$$

This is a self-contained equation for the evolution of $\mathcal{C}$.

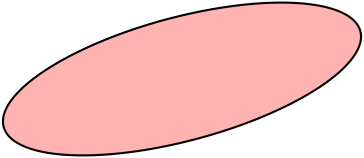
It is known as 'Contour Dynamics' (Zabusky, Hughes & Roberts 1979) or the 'Water Bag Model' (Berk & Roberts 1965).

Vorticity interfaces

Contour Dynamics in an infinite-order Hamiltonian dynamical system.

As a result, it is capable of exhibiting arbitrarily great complexity.

Many (equilibrium) solutions exist, some exact like the line, the circle and the ellipse (Kirchhoff 1876).



The line and circle have been proven to be linearly stable (Kelvin 1880), and the ellipse as well for aspect ratios less than 3 (Love 1893).

Furthermore, the line and circle have been proven to be nonlinearly stable (Dritschel, *JFM* **191**, 575–582, 1988).

Evolution of small disturbances

Lord Kelvin (William Thomson) in his paper "Vibrations of a columnar vortex" (*Phil. Mag.* **10**, 155–168, 1880) first considered the behaviour of *linear, small-amplitude waves*.

The equilibrium flow consists of a circular region of uniform vorticity surrounded by irrotational fluid extending to infinity. In 2D, this is called a circular "vortex patch".

In 2D, only boundary deformations matter for the flow evolution. One can therefore consider (linear) boundary waves of the form

$$r(\theta, t) = R + \Re\{\eta(\theta, t)\} \quad \text{where} \quad \eta(\theta, t) = a\, e^{\mathrm{i}(m\theta - \sigma t)}$$

where r is radius, R is the mean radius, $|a| \ll R$ is the disturbance amplitude, m is its azimuthal wavenumber and σ is its frequency. This *ansatz* takes advantage of the rotational symmetry of the equilibrium flow.

Linear behaviour

The frequency σ is determined as an eigenvalue, after expanding the governing equations to first order in $\eta/R \ll 1$, neglecting higher-order terms.

The most direct approach is to use the contour-dynamics equation, since then there are no boundary conditions to consider. After much algebra ... one may show

$$\frac{\partial \eta}{\partial t} + \tfrac{1}{2}\Delta\omega \frac{\partial \eta}{\partial \theta} = \frac{\Delta\omega}{4\pi} \int_0^{2\pi} \frac{\eta(\alpha, t)\,\sin(\alpha - \theta)}{1 - \cos(\alpha - \theta)}\, \mathrm{d}\alpha$$

— to leading-order in η/R. This is *non-local.*

Of course, $\eta = 0$ is a solution (trivial), but there also exist non-trivial solutions. Inserting $\eta(\theta, t) = a\, e^{\mathrm{i}(m\theta - \sigma t)}$, one finds

$$-\mathrm{i}\sigma\,\eta = -\tfrac{1}{2}\Delta\omega\,\mathrm{i}\,(m-1)\,\eta\,.$$

Linear behaviour

For $\eta \neq 0$, we can cancel $\mathrm{i}\,\eta$ to obtain *the dispersion relation*

$$\sigma = \tfrac{1}{2}\Delta\omega\,(m-1)$$

(Kelvin, 1880). This does not appear to be special, but in equilibrium the boundary rotates at the rate $\frac{1}{2}\Delta\omega$. Thus, in a frame of reference rotating with the vortex boundary,

$$\sigma = -\tfrac{1}{2}\Delta\omega$$

— independent of m. *There is no dispersion!*

Hence, a disturbance made up of an arbitrary superposition of waves

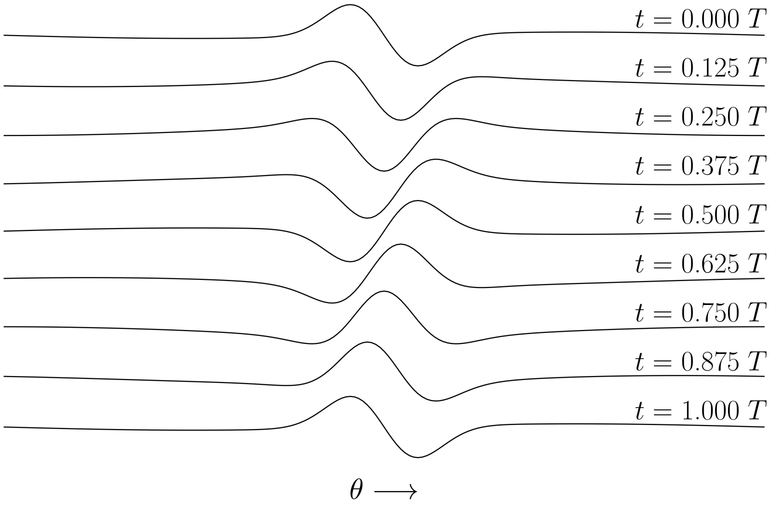
$$\eta(\theta,0) = \sum_{m>0} a_m\, e^{\mathrm{i}m\theta}$$

evolves, *in linear theory*, according to

$$\eta(\theta,t) = \eta(\theta,0)\, e^{\frac{1}{2}\mathrm{i}\Delta\omega t}\,.$$

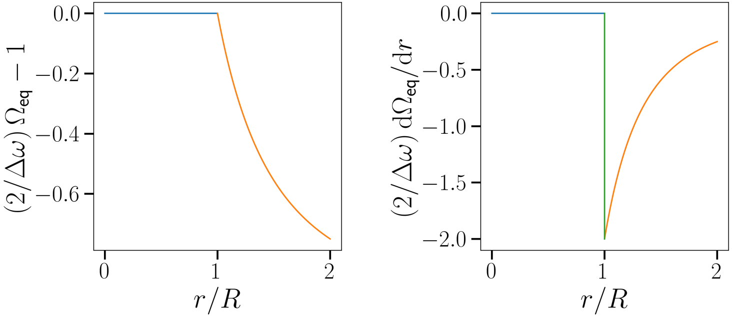
Linear behaviour — evolution over one period

Evolution of the real part of $\eta(\theta, t)$ over one period $T = 4\pi/\Delta\omega$



Nonlinear behaviour

Any *finite-amplitude* disturbance will experience differential rotation — shear — arising from the equilibrium flow. (Below, $\Omega = \mathrm{d}\theta/\mathrm{d}t$.)



Fluid particles at outward-pointing wave crests ($r > R$) will move, tangentially (in θ), slightly faster than particles at smaller radii.

Kelvin (1880) reasoned (correctly!) that this would eventually cause the wave to steepen and break.

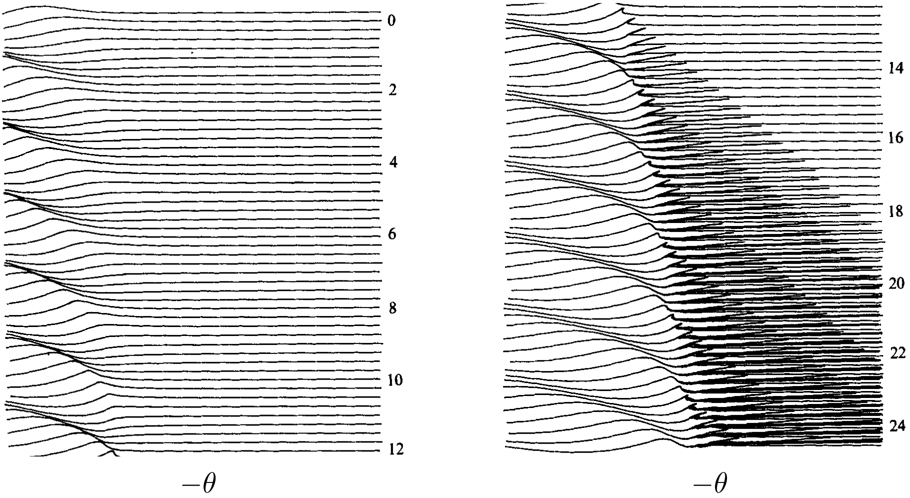
Some history

From Alex D. D. Craik "Lord Kelvin on fluid mechanics" (Eur. Phys. J. **H37**, 75–114, 2012):

On introducing a slight deformation, "the central vortex column is set into vibration" creating waves that cause "corrugation to travel round the cylindrical bounding sheet, by which energy is consumed, and moment of momentum taken out of the fluid".

... Kelvin asserts that: "The consumption of energy still goes on, and the way in which it goes on is this: the waves of shorter length are indefinitely multiplied and exalted till their crests run out into fine laminae of liquid.... Thus a certain portion of the irrotationally revolving water becomes mingled with the central vortex column. The process goes on until what may be called a vortex sponge is formed ... consisting of portions of rotational and irrotational fluid, more and more finely mixed together as time advances."

108 years later ... from Dritschel (*J. Fluid Mech.* **194**, 511–547, 1988):



"The repeated filamentation of two-dimensional vorticity interfaces"

Weakly nonlinear theory

In Dritschel (*JFM* **194**), a weakly nonlinear theory was developed to describe the observed wave steepening up to the onset of filamentation.

That theory predicts that filamentation will occur on a time-scale inversely proportional to the *square* of the maximum initial *wave slope* — for **any** unsteady disturbance.

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We sketch the derivation next and examine newly-discovered mathematical properties, including the self-similar blow-up of wave slope.

[This is joint work with Adrian Constantin (University of Vienna) and Pierre Germain (Imperial College London).]

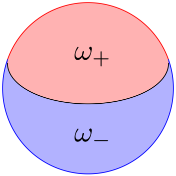
Weakly nonlinear theory

In Cartesian coordinates, the equations of Contour Dynamics on the surface of a sphere closely resemble those on the plane:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\frac{\Delta\omega}{2\pi}\int_{\mathcal{C}} \log|\boldsymbol{x}' - \boldsymbol{x}|\,\mathrm{d}\boldsymbol{x}'\,.$$

The only difference is that on the sphere, the points $\boldsymbol{x}$ and $\boldsymbol{x}'$ are *three-dimensional* but constrained to have unit magnitude, without loss of generality (Dritschel, *J. Comput. Phys.* **78**, 477–483, 1988).

The weakly nonlinear theory (WNT) was developed for a zonal circular patch, $z = z_0$, dividing the sphere into two regions with vorticity ω_+ to the north ($z > z_0$) and vorticity ω_- to the south ($z < z_0$).



The planar limit may be recovered by considering $z_0 \to 1$.

Weakly nonlinear theory

The requirement (by Stokes' theorem) that the integral of ω vanishes over any closed surface means

$$(1 - z_0)\omega_+ + (1 + z_0)\omega_- = 0 \,.$$

Together with the definition $\Delta\omega = \omega_+ - \omega_-$, we have

$$\omega_+ = \tfrac{1}{2}(1 + z_0)\Delta\omega \qquad \text{and} \qquad \omega_- = -\tfrac{1}{2}(1 - z_0)\Delta\omega$$

which implies that the jump $\Delta\omega$ alone is sufficient to specify ω over the whole sphere.

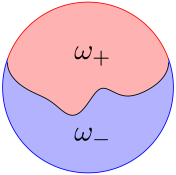
Note that the area of the region north of z_0 is $2\pi(1 - z_0)$, while the area of the region south of z_0 is $2\pi(1 + z_0)$.

In the planar limit, $\omega_+ \to \Delta\omega$ and $\omega_- \to 0$.

Weakly nonlinear theory

We consider a disturbed vorticity interface of the form

$$z(\theta, t) = z_0 - r_0^2\, \rho(\theta, t)$$



where $r_0 = (1 - z_0^2)^{\frac{1}{2}}$.

Here $\rho(\theta, t)$ is the displacement function, considered small compared to unity (this form facilitates taking the planar limit). We take

$$\rho(\theta, t) = a\, \eta(\theta, t)$$

where $a \ll 1$ and $\eta = \mathcal{O}(1)$.

Using the multiple-time-scales expansion in Dritschel (*JFM* **194**), we take

$$\eta(\theta, t) = \mathcal{A}(\theta, t) e^{\frac{1}{2}\mathrm{i}\Delta\omega\, t} + \mathrm{c.c.}$$

where $\mathcal{A}(\theta, t) = \mathcal{A}_0(\theta, \tau) + a\, \mathcal{A}_1(\theta, t, \tau) + a^2 \ldots$, and where $\tau = \Delta\omega\, a^2 t$ is the *slow time*.

Weakly nonlinear theory

Plugging this into the Contour Dynamics equation, expanding to $\mathcal{O}(a^3)$, and removing any terms that would lead to secular growth, one finds a cubically-nonlinear equation for $\mathcal{A}_0$ (hereafter written simply as $\mathcal{A}$):

$$\frac{\partial \mathcal{A}}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial \theta}\left[z_0^2 \mathcal{T}_1 + \mathcal{T}_2 - (z_0^2+1)\,|\mathcal{A}|^2 \mathcal{A}\right]$$

where

$$\mathcal{T}_1 = \mathrm{i}\left(\mathcal{A}\frac{\partial}{\partial \theta}(\mathcal{W} - \overline{\mathcal{W}}) - |\mathcal{A}|^2 \frac{\partial \mathcal{A}}{\partial \theta}\right)$$

and

$$\mathcal{T}_2 = -\frac{1}{4\pi}\int_0^{2\pi} \frac{|\mathcal{A}(\alpha,\tau) - \mathcal{A}(\theta,\tau)|^2\left[\mathcal{A}(\alpha,\tau) - \mathcal{A}(\theta,\tau)\right]}{1-\cos(\alpha-\theta)}\,\mathrm{d}\alpha\,.$$

Above, $\mathcal{W}$ is the part of $|\mathcal{A}|^2$ expressible in positive wavenumbers, i.e.

$$\mathcal{W} = \sum_{m=1}^{\infty} w_m(\tau)\, e^{\mathrm{i}m\theta}\,,$$

and an over-bar denotes complex conjugation.

Weakly nonlinear theory

Surprisingly, by explicit calculation, starting with the general Fourier series

$$\mathcal{A} = \sum_{m=1}^{\infty} a_m e^{\mathrm{i}m\theta} \,,$$

one can show $\mathcal{T}_2 = \mathcal{T}_1$ **!** In this case, we have more simply

$$\frac{\partial \mathcal{A}}{\partial \tilde{\tau}} = \frac{\partial \mathcal{B}}{\partial \theta}$$

where $\tilde{\tau} = \frac{1}{2}(z_0^2 + 1)\,\tau$ is a re-scaled slow time, and $\mathcal{B} = \mathcal{T}_2 - |\mathcal{A}|^2\mathcal{A}$.
Explicitly,

$$\mathcal{B}(\theta, \tilde{\tau}) = -\frac{1}{4\pi}\int_0^{2\pi} \frac{|\mathcal{A}(\alpha,\tilde{\tau}) - \mathcal{A}(\theta,\tilde{\tau})|^2\,[\mathcal{A}(\alpha,\tilde{\tau}) - \mathcal{A}(\theta,\tilde{\tau})]}{1-\cos(\alpha-\theta)}\,\mathrm{d}\alpha - |\mathcal{A}(\theta,\tilde{\tau})|^2\mathcal{A}(\theta,\tilde{\tau})\,.$$

Weakly nonlinear theory

All explicit dependence on z_0 *disappears*! This is a universal equation for the onset of filamentation.

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In spectral form, the Fourier coefficients $a_m(\tilde{\tau})$ of $\mathcal{A}$ obey

$$\frac{\mathrm{d}a_m}{\mathrm{d}\tilde{\tau}} = \tfrac{1}{2}\mathrm{i}m\sum_n\sum_p(n+p-|n-m|-|p-m|-2)a_n a_p \bar{a}_{n+p-m}\,,$$

where n and p are positive integers for which $n+p>m$.

The -2 in the brackets above comes from the $|\mathcal{A}|^2\mathcal{A}$ term in $\mathcal{B}$, a term which is absent for a periodic linear vorticity interface on the plane.

Weakly nonlinear theory

Some interesting properties of this equation:

- a_1 is constant $(\mathrm{d}a_1/\mathrm{d}\tilde{\tau} = 0)$;
- The 'momentum' $\mathcal{P} = \sum_m |a_m|^2$ is constant;
- The 'mass' $\mathcal{M} = \sum_m |a_m|^2/m$ is constant;
- The 'energy' $\mathcal{E}$ is constant.

The energy (kinetic energy) is

$$\mathcal{E} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{|\mathcal{A}(\alpha,\tilde{\tau}) - \mathcal{A}(\theta,\tilde{\tau})|^4}{1-\cos(\alpha-\theta)}\,\mathrm{d}\alpha\,\mathrm{d}\theta - \frac{2}{\pi}\int_0^{2\pi} |\mathcal{A}(\theta,\tilde{\tau})|^4\,\mathrm{d}\theta\,.$$

Further mathematical results can be found in Constantin, Dritschel & Germain, *Nonlinearity* (2024).

Results

In the WNT, all disturbances consisting of a *single harmonic* $m = k$ are periodic in time, and propagate at speed $\mathrm{d}\theta/\mathrm{d}\tilde{\tau} = -k(k-1)|a_k|^2$.

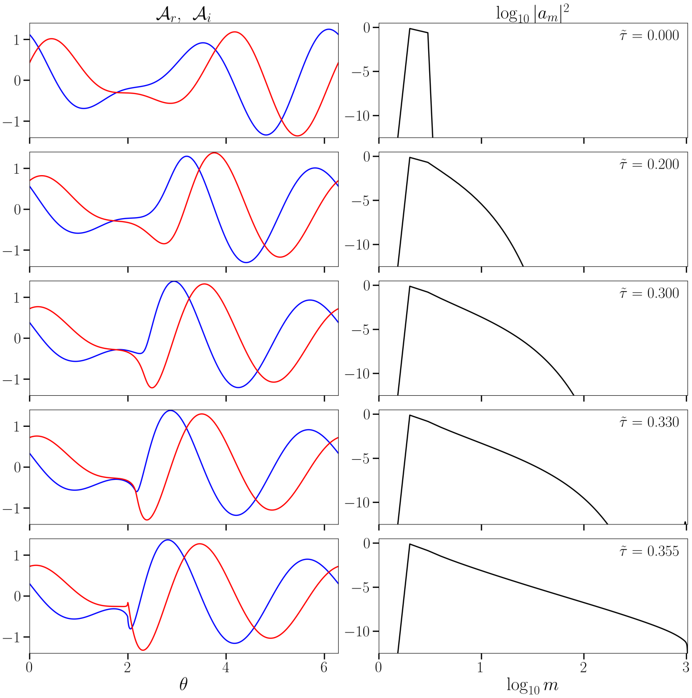
Consider the initial condition

$$\mathcal{A}(\theta, 0) = a_k(0)e^{\mathrm{i}k\theta} + a_p(0)e^{\mathrm{i}p\theta}$$

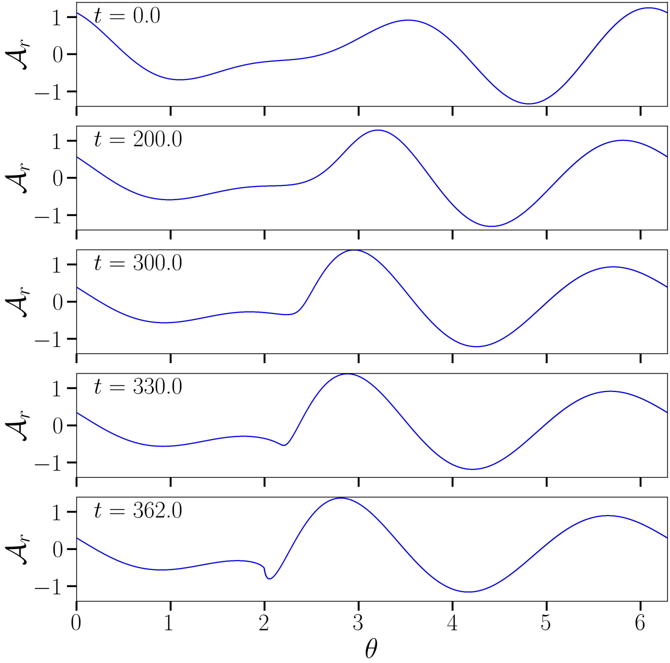
for $p > k$. Here, we take $a_p(0) = \epsilon e^{\mathrm{i}\pi/3}$ and $a_k(0) = (1-\epsilon^2)^{\frac{1}{2}}$, with $k = 2$ and $p = 3$.

The first example uses $\epsilon = 0.5$. [movie]

Numerically, we solve for the Fourier modes explicitly but truncating m to $M = 2048$ modes. The invariants are monitored to ensure accuracy.

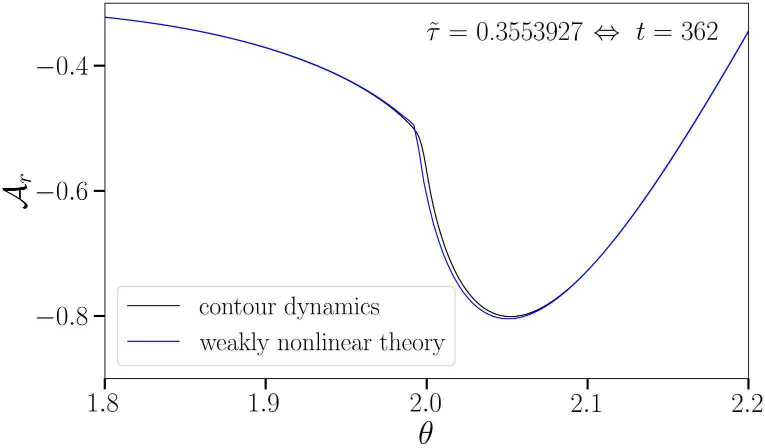


Real and imaginary parts of $\mathcal{A}$ ($\mathcal{A}_r$ and $\mathcal{A}_i$) are shown in blue and red respectively.



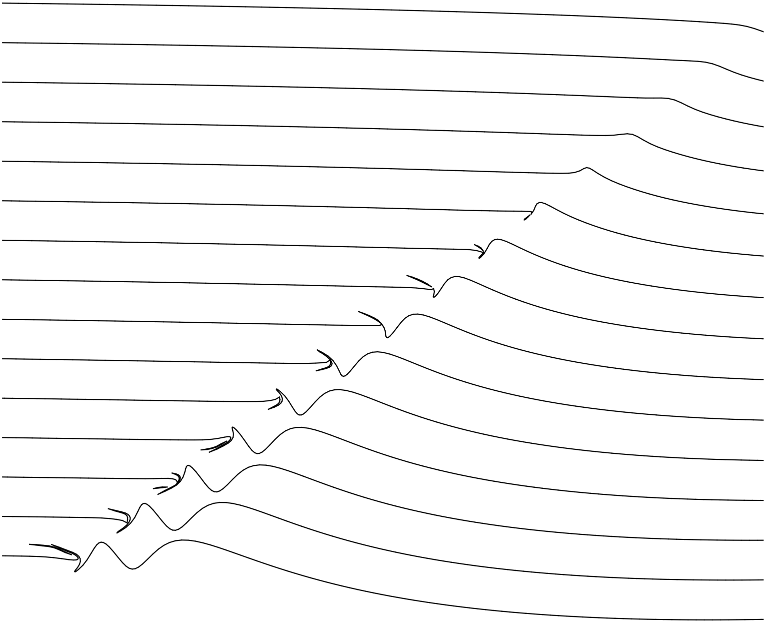
Real part of $\mathcal{A}$, $\mathcal{A}_r$, *compared* with a spherical Contour Dynamics simulation starting with $\rho = 2a\mathcal{A}_r$ and $a = 1/40$.

Upon closer inspection...

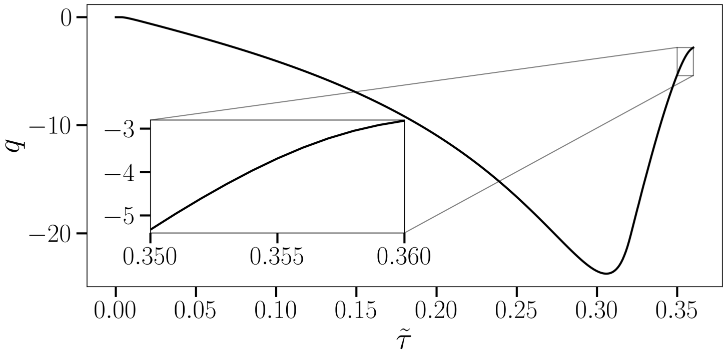


The WNT describes the onset of filamentation exceedingly well.

And what happens later ... in Contour Dynamics?



Evidence for a finite-time singularity in the WNT

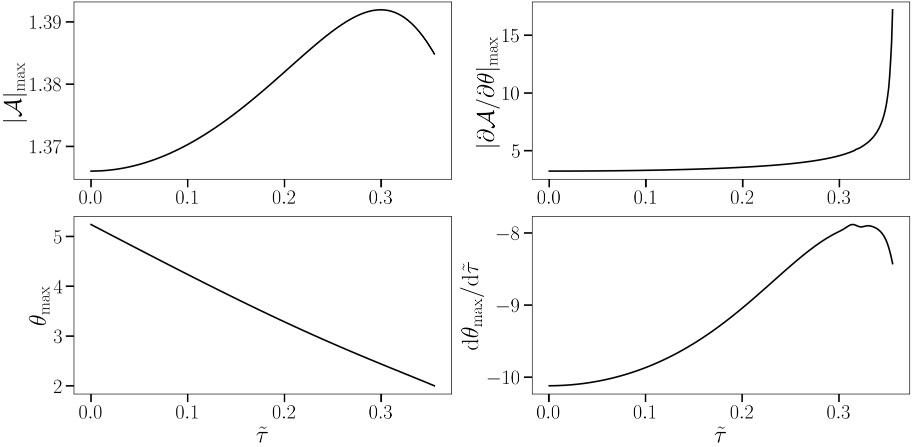


Evolution of the spectral slope q, obtained by a least-squares fit of $\log|a_m|^2$ to $q\log m + c$, between wavenumbers $m = 10$ and $M/2 = 1024$.

The slope appears to shallow to a critical value of -3

Evidence for a finite-time singularity in the WNT

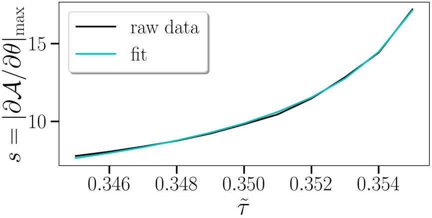
Maximum amplitude — Maximum wave slope s

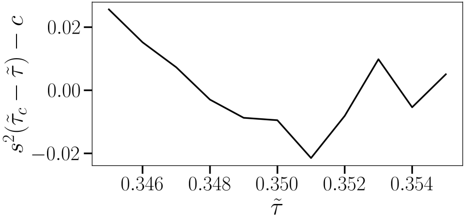


Location of $s_{\max}$ — Corresponding speed

The maximum wave slope appears to diverge in finite time

Evidence for a finite-time singularity in the WNT





Evolution of the maximum wave slope $s(\tilde{\tau}) = |\partial\mathcal{A}/\partial\theta|_{\max}$ together with a fit to $\sqrt{c/(\tilde{\tau}_c - \tilde{\tau})}$ (left), and the function $f(\tilde{\tau}) = s^2(\tilde{\tau}_c - \tilde{\tau}) - c$ (right) which would be zero for a perfect fit.

Self-similar blow up

The observations suggest looking for a solution of the form

$$\mathcal{A}(\theta, \tilde{\tau}) = \delta^{\mathrm{i}\mu} \psi(\xi)\,,$$

where ψ is some *universal* function, and where

$$\xi = \frac{\theta - \theta_{\max}(\tilde{\tau})}{\sqrt{\delta}} \quad \text{and} \quad \delta = 1 - \tilde{\tau}/\tilde{\tau}_c \ll 1\,.$$

Above, $\tilde{\tau}_c$ is the *singularity time,* to be determined along with $\psi(\xi)$.

The $\delta^{\mathrm{i}\mu}$ pre-factor (with μ real) is the most general form consistent with $|\mathcal{A}|$ remaining finite and non-zero as $\tilde{\tau} \to \tilde{\tau}_c$.

As observed, this form leads to

$$\left|\frac{\partial \mathcal{A}}{\partial \theta}\right|_{\max} \propto \frac{1}{\sqrt{\tilde{\tau}_c - \tilde{\tau}}}\,.$$

Self-similar blow up: the universal equation

Derivation (abbreviated)

Take $\theta = \theta_{\max} + \delta^{\frac{1}{2}}\xi$ and $\alpha = \theta_{\max} + \delta^{\frac{1}{2}}\xi'$. For $\delta \ll 1$, assuming $\xi' - \xi = \mathcal{O}(1)$, one can show that

$$\mathcal{B} = \delta^{\mathrm{i}\mu - \frac{1}{2}} b(\xi)$$

to leading order in δ, where

$$b(\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\psi(\xi') - \psi(\xi)|^2\,[\psi(\xi') - \psi(\xi)]}{(\xi' - \xi)^2}\,\mathrm{d}\xi'$$

assuming sufficiently fast far-field decay.

Note: the neglected term, $|\mathcal{A}|^2\mathcal{A}$, is $\mathcal{O}(\delta^{\frac{1}{2}})$ smaller.

Self-similar blow up: the universal equation

The left-hand side of the WNT equation evaluates to

$$\frac{\partial \mathcal{A}}{\partial \tilde{\tau}} = \frac{\partial}{\partial \tilde{\tau}}\left(\delta^{\mathrm{i}\mu}\psi(\xi)\right) = \mathrm{i}\mu\delta^{\mathrm{i}\mu-1}\psi + \delta^{\mathrm{i}\mu}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\frac{\partial \xi}{\partial \tilde{\tau}}\,.$$

However, since $\xi = (\theta - \theta_{\mathsf{max}})\delta^{-\frac{1}{2}}$ and $\delta = 1 - \tilde{\tau}/\tilde{\tau}_c$, we have

$$\frac{\partial \xi}{\partial \tilde{\tau}} = -\delta^{-\frac{1}{2}}\frac{\mathrm{d}\theta_{\mathsf{max}}}{\mathrm{d}\tilde{\tau}} + \delta^{-1}\frac{\xi}{2\tilde{\tau}_c}\,.$$

For $\delta \ll 1$, we can neglect the $\mathcal{O}(\delta^{-\frac{1}{2}})$ term, giving

$$\frac{\partial \mathcal{A}}{\partial \tilde{\tau}} \approx \delta^{\mathrm{i}\mu-1}\left(\mathrm{i}\mu\psi + \frac{\xi}{2\tilde{\tau}_c}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)$$

to leading order in δ.

Self-similar blow up: the universal equation

The right-hand side of the WNT equation evaluates to

$$\frac{\partial \mathcal{B}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\delta^{\mathrm{i}\mu-\frac{1}{2}}\, b(\xi)\right) = \delta^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\delta^{\mathrm{i}\mu-\frac{1}{2}}\, b(\xi)\right) = \delta^{\mathrm{i}\mu-1}\frac{\mathrm{d}b}{\mathrm{d}\xi}\,.$$

Cancelling the factor $\delta^{\mathrm{i}\mu-1}$ from the l.h.s. and r.h.s., we find

$$\mathrm{i}\mu\psi + \frac{\xi}{2\tilde{\tau}_c}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = -\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_{-\infty}^{\infty}\frac{|\psi(\xi')-\psi(\xi)|^2\,[\psi(\xi')-\psi(\xi)]}{(\xi'-\xi)^2}\,\mathrm{d}\xi'\,.$$

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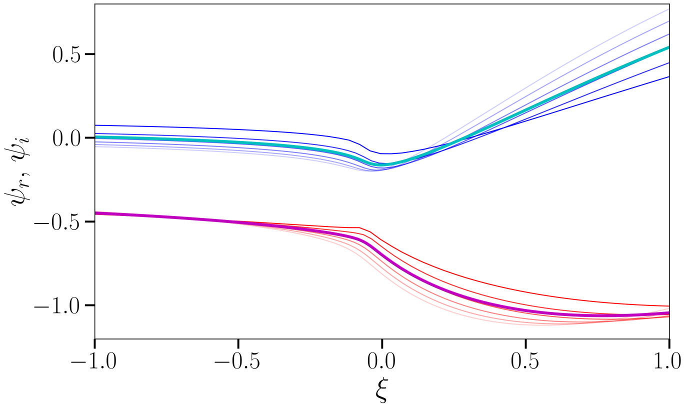
The claim is that this equation describes the blow up of wave slope on vorticity interfaces, and that almost all initial conditions are attracted to this blow up solution in finite time.

Fit to the self-similar form



The estimated self-similar solution $\psi(\xi)$ (real part in cyan, imaginary part in magenta), together with scaled numerical profiles, $\delta^{-\mathrm{i}\mu}\mathcal{A}(\theta_{\max} + \delta^{\frac{1}{2}}\xi, \tilde{\tau})$, at times $\tilde{\tau} = 0.345$, 0.347, 0.349, 0.351, 0.353 and 0.355 (real part in blue, imaginary part in red, with fading backwards in time).

Exact solutions: stability

We noted earlier that the single-mode solutions

$$\mathcal{A}(\theta,\tilde{\tau}) = a_k e^{\mathrm{i}k\theta + k(k-1)|a_k|^2\tilde{\tau}}$$

are *exact*, translating wave solutions of the WNT.

There are the analogues of the "V-state" solutions first discovered by Deem & Zabusky, *Phys. Rev. Lett.* **40(13)**, 859–862 (1978).

We have also discovered *exact*, periodic-in-time two-mode solutions [movie]:

$$\mathcal{A}(\theta,\tilde{\tau}) = a_1 e^{\mathrm{i}\theta} + a_k e^{\mathrm{i}k\theta + k(k-1)(2|a_1|^2+|a_k|^2)\tilde{\tau}} .$$

We consider next the stability of these solutions. Does a finite-time singularity in wave slope still occur? Is it inevitable?

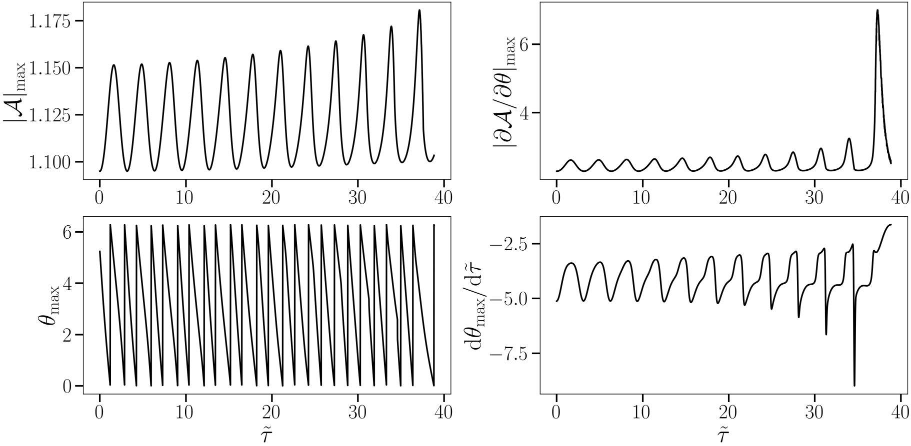
Consider the same form of initial condition as previously:

$$\mathcal{A}(\theta, 0) = (1-\epsilon^2)^{\frac{1}{2}} e^{2i\theta} + \epsilon e^{i\pi/3} e^{3i\theta}$$

but now with $\epsilon = 0.1$ (5 times smaller). [movie]

Maximum amplitude

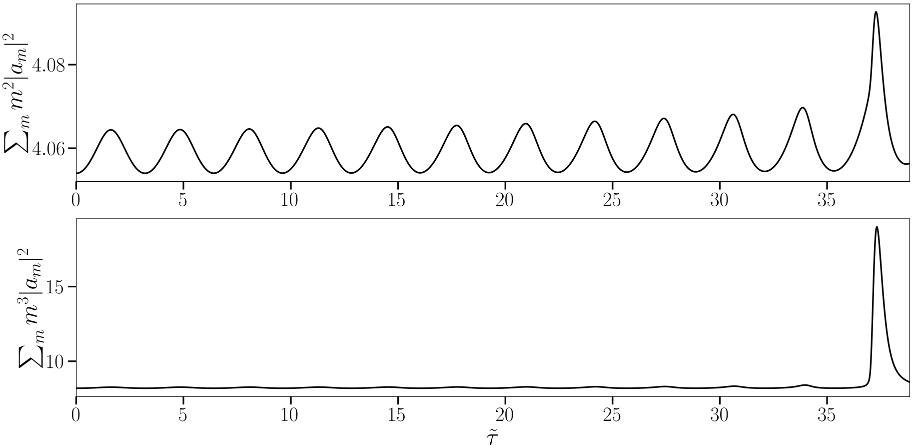
Maximum wave slope s



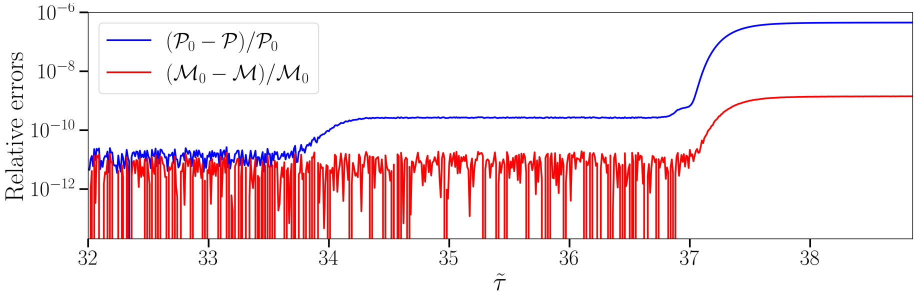
Location of $s_{\max}$

Corresponding speed

Evolution of spectral norms



Accuracy? Conservation of momentum $\mathcal{P}$ and mass $\mathcal{M}$



Note: the spectral coefficients a_m were saved to 11 decimal places, so the error before $t = 34$ is even less than that shown.

The simulation becomes unreliable after $t = 37$ due to the numerical truncation and filtering of high m modes.

• *Filamentation likely occurs just after $t = 37$* •

Conclusions

- Two-dimensional flows described by Euler's equation admit a Contour Dynamics formulation for piecewise-uniform vorticity.

- A circular or linear interface is an *exact*, steady solution of the system.

- Small disturbances, in a frame of reference moving with the mean velocity of the interface, simply oscillate in place with no dispersion (in linear theory).

- Nonlinearity allows for a progressive, inevitable steepening, resulting in repeated filamentation, and an increasingly complex vorticity interface.

- A weakly nonlinear theory, originally developed in Dritschel (*JFM* **194**, 1988) accurately describes the onset of filamentation.