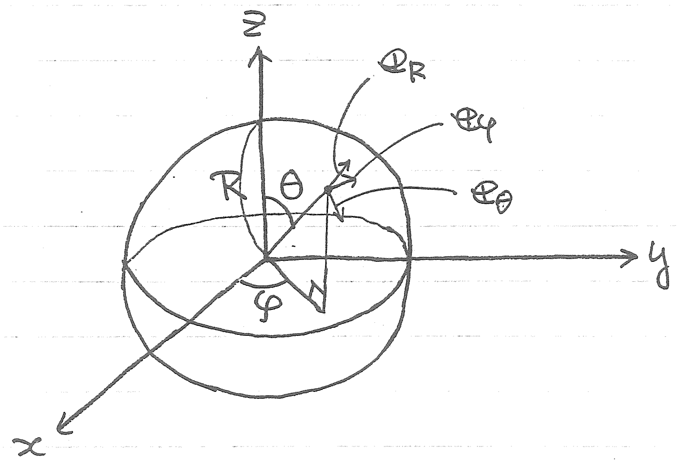
Notations and Fundamentals

座標



単位ベクトル

$$\mathbb{e}_R = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$
$$\mathbb{e}_\theta = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)$$
$$\mathbb{e}_\varphi = (-\sin\varphi,\ \cos\varphi,\ 0)$$

単位ベクトルの微分

$$\frac{\partial}{\partial R}\begin{pmatrix}\mathbb{e}_R\\ \mathbb{e}_\theta\\ \mathbb{e}_\varphi\end{pmatrix} = 0$$

$$\frac{\partial}{\partial \theta}\begin{pmatrix}\mathbb{e}_R\\ \mathbb{e}_\theta\\ \mathbb{e}_\varphi\end{pmatrix} = \begin{pmatrix}\mathbb{e}_\theta\\ -\mathbb{e}_R\\ 0\end{pmatrix}$$

$$\frac{\partial}{\partial \varphi}\begin{pmatrix}\mathbb{e}_R\\ \mathbb{e}_\theta\\ \mathbb{e}_\varphi\end{pmatrix} = \begin{pmatrix}\sin\theta\,\mathbb{e}_\varphi\\ \cos\theta\,\mathbb{e}_\varphi\\ -\sin\theta\,\mathbb{e}_R - \cos\theta\,\mathbb{e}_\theta\end{pmatrix}$$

微分演算子

- $$\nabla = \mathbb{e}_R \frac{\partial}{\partial R} + \mathbb{e}_\theta \frac{\partial}{R\partial\theta} + \mathbb{e}_\varphi \frac{\partial}{R\sin\theta\partial\varphi}$$ ▨

- $$\nabla\cdot\mathbb{u} = \left(\mathbb{e}_R \frac{\partial}{\partial R} + \mathbb{e}_\theta \frac{1}{R}\frac{\partial}{\partial\theta} + \mathbb{e}_\varphi \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(u_R\mathbb{e}_R + u_\theta\mathbb{e}_\theta + u_\varphi\mathbb{e}_\varphi\right)$$
$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}u_\varphi$$ ▨

- $$\nabla\times\mathbb{u} = \left(\mathbb{e}_R \frac{\partial}{\partial R} + \mathbb{e}_\theta \frac{1}{R}\frac{\partial}{\partial\theta} + \mathbb{e}_\varphi \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\left(u_R\mathbb{e}_R + u_\theta\mathbb{e}_\theta + u_\varphi\mathbb{e}_\varphi\right)$$
$$= \mathbb{e}_R \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\varphi) - \frac{\partial}{\partial\varphi}u_\theta\right]$$
$$+ \mathbb{e}_\theta \frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}u_R - \frac{\partial}{\partial R}(R u_\varphi)\right]$$
$$+ \mathbb{e}_\varphi \frac{1}{R}\left[\frac{\partial}{\partial R}(R u_\theta) - \frac{\partial}{\partial\theta}u_R\right]$$ ▨

- $$\Delta = \nabla\cdot\nabla = \frac{1}{R^2}\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$= \frac{1}{R^2}\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R} + \frac{1}{R^2}\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\varphi^2}\right]$$
$(\mu = \cos\theta)$ ▨

球面積分

$$\int dS = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi R^2 \sin\theta = \int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi R^2$$

部分積分 (A, B は球面上の smooth な関数)

- $\int A \Delta B dS = -\int \nabla A \cdot \nabla B dS = \int (\Delta A) B dS$
- $\int A \frac{\partial B}{\partial \varphi} dS = -\int \frac{\partial A}{\partial \varphi} B dS$

Jacobian の積分

$$\int \frac{1}{\sin\theta} \frac{\partial(A, B)}{\partial(\theta, \varphi)} dS = \int \frac{\partial(A, B)}{\partial(\mu, \varphi)} dS = 0$$

・運動方程式（Euler方程式）

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u} + \nabla\frac{u^2}{2} = -\frac{1}{\rho}\nabla p$$

極座標表示

$$\begin{cases} \dfrac{\partial}{\partial t}u_R + (\boldsymbol{u}\cdot\nabla)u_R - \dfrac{1}{R}(u_\theta^2+u_\varphi^2) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial R} \\ \dfrac{\partial}{\partial t}u_\theta + (\boldsymbol{u}\cdot\nabla)u_\theta + \dfrac{1}{R}(u_R u_\theta - u_\varphi^2\cot\theta) = -\dfrac{1}{\rho}\dfrac{1}{R}\dfrac{\partial p}{\partial \theta} \\ \dfrac{\partial}{\partial t}u_\varphi + (\boldsymbol{u}\cdot\nabla)u_\varphi + \dfrac{1}{R}(u_R u_\varphi + u_\theta u_\varphi\cot\theta) = -\dfrac{1}{\rho}\dfrac{1}{R\sin\theta}\dfrac{\partial p}{\partial \varphi} \end{cases}$$

$$\left(\boldsymbol{u}\cdot\nabla = u_R\frac{\partial}{\partial R} + u_\theta\frac{1}{R}\frac{\partial}{\partial\theta} + u_\varphi\frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

球面上 $\left(u_R = \frac{\partial}{\partial R} = 0\right)$

・$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = \omega_R \boldsymbol{e}_R$

$$\omega_R = \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\varphi) - \frac{\partial}{\partial\varphi}u_\theta\right]$$

非発散 （ρ = 定数）

$$\nabla\cdot\boldsymbol{u} = \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{\partial u_\varphi}{\partial\varphi}\right] = 0$$

流れ関数 ψ

$$\begin{pmatrix} u_\theta \\ u_\varphi \end{pmatrix} = \begin{pmatrix} \dfrac{1}{R\sin\theta}\dfrac{\partial\psi}{\partial\varphi} \\ -\dfrac{1}{R}\dfrac{\partial\psi}{\partial\theta} \end{pmatrix}$$

渦度方程式

$$\frac{\partial}{\partial t}\omega_R + (\boldsymbol{u}\cdot\nabla)\omega_R = 0$$

where

- $$(\boldsymbol{u}\cdot\nabla)\omega_R = u_\theta \frac{1}{R}\frac{\partial}{\partial\theta}\omega_R + u_\varphi \frac{1}{R}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\omega_R$$
$$= \frac{1}{R^2}\frac{1}{\sin\theta}\frac{\partial(\omega_R, \psi)}{\partial(\theta, \varphi)} = -\frac{1}{R^2}\frac{\partial(\omega_R, \psi)}{\partial(\mu, \varphi)}$$
$(\mu = \cos\theta)$

- $$\omega_R = -\Delta\psi$$
$$= -\frac{1}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]$$
$$= -\frac{1}{R^2}\left[\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\psi}{\partial\mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\varphi^2}\right]$$

Colioris項

$f = 2\Omega$

$$\mathbb{f} = f\mathbb{e}_z = (\cos\theta\,\mathbb{e}_R - \sin\theta\,\mathbb{e}_\theta) f$$

$$\mathbb{f}\times\mathbb{u} = f\mathbb{e}_z \times (u_\theta\mathbb{e}_\theta + u_\varphi\mathbb{e}_\varphi)$$
$$= (-\sin\theta\, u_\varphi\mathbb{e}_R - \cos\theta\, u_\varphi\mathbb{e}_\theta + \cos\theta\, u_\theta\mathbb{e}_\varphi) f$$

球面に沿う成分だけを残す.

$$\mathbb{f}\times\mathbb{u} = (-\cos\theta\, u_\varphi\mathbb{e}_\theta + \cos\theta\, u_\theta\mathbb{e}_\varphi) f$$

$$\nabla\times(\mathbb{f}\times\mathbb{u}) = -\frac{1}{R} f\sin\theta\, u_\theta\,\mathbb{e}_R = -\frac{f}{R^2}\frac{\partial\psi}{\partial\varphi}\mathbb{e}_R$$

回転系の渦度方程式

$$\frac{\partial}{\partial t}\omega_R + \frac{1}{R^2}\frac{1}{\sin\theta}\frac{\partial(\omega_R, \psi)}{\partial(\theta, \varphi)} - \frac{1}{R^2} f\frac{\partial\psi}{\partial\varphi} = 0$$

又は

$$\frac{\partial}{\partial t}\omega_R - \frac{1}{R^2}\frac{\partial(\omega_R, \psi)}{\partial(\mu, \varphi)} - \frac{1}{R^2} f\frac{\partial\psi}{\partial\varphi} = 0$$

$(\mu = \cos\theta)$

保存量

1. $\int \omega_R dS$

$$\frac{\partial}{\partial t}\int \omega_R dS = \frac{1}{R^2}\int dS\left[\frac{\partial(\omega_R,\psi)}{\partial(\mu,\varphi)} + f\frac{\partial\psi}{\partial\varphi}\right] = 0$$

2. $\int \omega^2 dS = \int (\nabla\psi)^2 dS$

$$\frac{\partial}{\partial t}\int (\nabla\psi)^2 dS = -2\int \psi\frac{\partial}{\partial t}\omega_R dS$$
$$= -\frac{2}{R^2}\int dS\left[\psi\frac{\partial(\omega_R,\psi)}{\partial(\mu,\varphi)} + \psi f\frac{\partial\psi}{\partial\varphi}\right]$$
$$= -\frac{1}{R^2}\int dS\left[\frac{\partial(\omega_R,\psi^2)}{\partial(\mu,\varphi)} + f\frac{\partial}{\partial\varphi}\psi^2\right] = 0$$

3. $\int \omega_R^2 dS$

$$\frac{\partial}{\partial t}\int \omega_R^2 dS = 2\int \omega_R\frac{\partial}{\partial t}\omega_R dS$$
$$= \int dS\left[\frac{\partial(\omega_R^2,\psi)}{\partial(\mu,\varphi)} + 2f\omega_R\frac{\partial\psi}{\partial\varphi}\right]$$
$$= -2f\int(\Delta\psi)\frac{\partial\psi}{\partial\varphi}dS = 2f\int\frac{\partial\psi}{\partial\varphi}(\Delta\psi)dS = 0$$

4. 角運動量のz成分

角運動量 $\mathbb{L}$

$$\mathbb{L} = \int \mathbb{r} \times \mathbb{p}\, dS = \rho R \int dS\, \mathbb{e}_R \times \mathbb{u} = \rho R \int dS \left[u_\theta \mathbb{e}_\varphi - u_\varphi \mathbb{e}_\theta \right]$$

$$= \rho R \int dS \begin{bmatrix} -u_\theta \sin\varphi - u_\varphi \cos\theta \cos\varphi \\ u_\theta \cos\varphi - u_\varphi \cos\theta \sin\varphi \\ u_\varphi \sin\theta \end{bmatrix}$$

$$= \rho \int dS \begin{bmatrix} 2\psi \sin\theta \cos\varphi \\ 2\psi \sin\theta \sin\varphi \\ 2\psi \cos\theta \end{bmatrix}$$

(ρ : 流体密度(定数))

$$\frac{\partial}{\partial t} L_z = \frac{\partial}{\partial t} 2\rho \int dS\, \psi \cos\theta$$

$$= 2\rho \int dS \frac{\partial \psi}{\partial t} \Delta\left(-\frac{R^2}{2}\cos\theta\right) \qquad \left(\because \Delta\left(-\frac{R^2}{2}\cos\theta\right) = \cos\theta\right)$$

$$= \rho R^2 \int dS \frac{\partial \omega_R}{\partial t} \cos\theta$$

$$= \rho R^2 \int dS \left[\mu \frac{\partial(\omega_R, \psi)}{\partial(\mu, \varphi)} + \mu f \frac{\partial \psi}{\partial \varphi} \right]$$

$$= \rho R^2 \int dS \left[\frac{\partial(\omega_R, \mu\psi)}{\partial(\mu, \varphi)} + \psi \frac{\partial}{\partial \varphi} \omega_R \right]$$

$$= -\rho R^2 \int dS\, \psi \frac{\partial}{\partial \varphi} \Delta \psi = \rho R^2 \int dS \left(\frac{\partial}{\partial \varphi} \Delta \psi\right) \psi = 0$$

(Note)

1. $\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_l^m Y_l^m$ の時

where

$Y_l^m = Y_l^m(\theta, \varphi)$ は球面調和関数

規格化は

$$\int dS\, Y_l^m Y_{l'}^{m'} = \delta_{ll'} \delta_{mm'} R^2$$

$$\mathbb{L} = \begin{bmatrix} 2\rho R^2 \sqrt{\frac{4\pi}{3}} \psi_1^0 \\ -\rho R^2 \sqrt{\frac{8\pi}{3}} (\psi_1^1 + \psi_1^{-1}) \\ -\rho R^2 \sqrt{\frac{8\pi}{3}} \frac{1}{i} (\psi_1^1 - \psi_1^{-1}) \end{bmatrix}$$

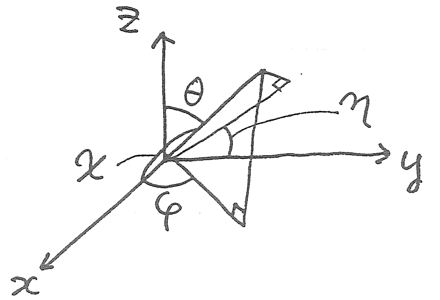
2. $f = 0$ の時 L_x, L_y, L_z が保存.

$$\frac{\partial}{\partial t} L_x = \rho \int dS \frac{\partial \psi}{\partial t} 2 \sin\theta \cos\varphi$$

$$= \rho \int dS \frac{\partial \psi}{\partial t} \Delta(-R^2 \sin\theta \cos\varphi)$$

$$= +\rho R^2 \int dS \frac{\partial \omega_R}{\partial t} \sin\theta \cos\varphi$$

この積分を計算するために座標変換 $(\theta, \varphi) \to (\chi, \eta)$



$$\begin{cases} \cos\chi = \sin\theta \cos\varphi & ① \\ \sin\chi \cos\eta = \sin\theta \sin\varphi & ② \\ \sin\chi \sin\eta = \cos\theta & ③ \end{cases}$$

①.②から.

$$\frac{\partial \chi}{\partial \theta} = -\frac{1}{\sin\chi}\cos\theta\cos\varphi \quad , \quad \frac{\partial \chi}{\partial \varphi} = \frac{1}{\sin\chi}\sin\theta\sin\varphi$$

$$\frac{\partial \eta}{\partial \theta} = -\frac{1}{\sin\chi\sin\eta}\left[\cos\theta\sin\varphi + \cot\chi\cos\eta\cos\theta\cos\varphi\right]$$

$$\frac{\partial \eta}{\partial \varphi} = -\cot\chi\sin\eta$$

これらを用いて

$$\frac{\partial(\chi.\eta)}{\partial(\theta.\varphi)} = \frac{\sin\theta\cos\theta}{\sin^2\chi\sin\eta} = \frac{\sin\theta}{\sin\chi} \quad (\because ③)$$

$$\therefore \int dS = \int d\theta d\varphi R^2 \sin\theta = \int d\chi d\eta R^2 \sin\chi$$

そこで

$$\frac{\partial}{\partial t}L_x = +\rho R^2 \int dS \frac{1}{R^2\sin\theta}\frac{\partial(\omega_R.4)}{\partial(\theta.\varphi)}\cos\chi$$

$$= +\rho R^2 \int dS \frac{1}{R^2\sin\theta}\frac{\partial(\chi.\eta)}{\partial(\theta.\varphi)}\frac{\partial(\omega_R.4)}{\partial(\chi.\eta)}\cos\chi$$

$$= +\rho R^2 \int dS \frac{1}{R^2\sin\chi}\frac{\partial(\omega_R.4)}{\partial(\chi.\eta)}\cos\chi = 0$$

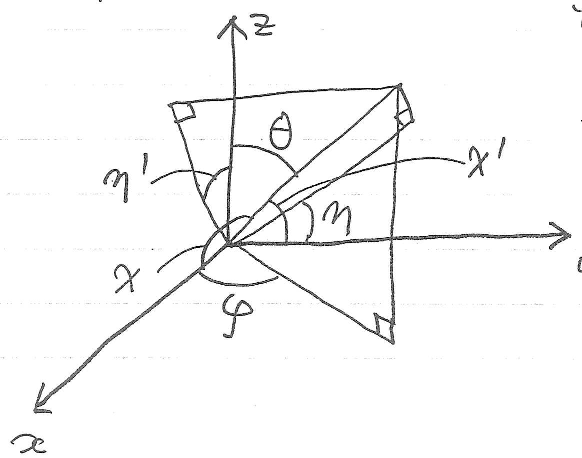
($\because$ L_zの保存と同じ計算)

$\frac{\partial}{\partial t}L_y$については. $(\theta.\varphi) \rightarrow (\chi.\eta) \rightarrow (\chi'.\eta')$の順に変数変換すれば

$$\sin\theta\sin\varphi = \sin\chi\cos\eta = \cos\chi'$$

を用いて $\frac{\partial}{\partial t}L_x$と同じ計算に帰着される。

$$\therefore \frac{\partial}{\partial t}L_y = 0$$



3. 絶対渦度の保存

渦度方程式

$$\frac{\partial \omega_R}{\partial t} - \frac{1}{R^2}\frac{\partial(\omega_R, \psi)}{\partial(\mu, \varphi)} - \frac{f}{R^2}\frac{\partial \psi}{\partial \varphi}$$

$$= \frac{\partial}{\partial t}(\omega_R + f\mu) - \frac{1}{R^2}\frac{\partial(\omega_R + f\mu, \psi)}{\partial(\mu, \varphi)}$$

$$= \left[\frac{\partial}{\partial t} + (u \cdot \nabla)\right](\omega_R + f\mu) = 0$$

thus $\int (\omega_R + f\mu)^2 ds$ は保存量.

但し

$$\int (\omega_R + f\mu)^2 ds = \int \omega_R^2 ds + 2f\int \omega_R \mu ds + \int \mu^2 ds$$

$$= \int \omega_R^2 ds + \frac{2f}{\rho R^2} L_z + \frac{4\pi}{3} R^2$$

だから独立な保存量ではない.

$\boxed{\text{平面上のStrain}}$

$$\nabla u = \begin{bmatrix} \frac{\partial u_x}{\partial x}, & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y}, & \frac{\partial u_y}{\partial y} \end{bmatrix}$$

固有値 λ

$$\lambda^2 - tr[\nabla u]\lambda + det[\nabla u] = 0$$

$$\lambda = \frac{1}{2}\left[tr[\nabla u] \pm \sqrt{(tr[\nabla u])^2 - 4det[\nabla u]} \right]$$

thus.

$$D = (tr[\nabla u])^2 - 4det[\nabla u]$$

$D > 0 \rightarrow$ 伸縮部分 \
$D < 0 \rightarrow$ 渦部分 } と定義

特に $\nabla \cdot u = tr[\nabla u] = 0$ ならば

$$D = -4det[\nabla u] = 4\left[\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} \right]$$

$\left(-\frac{1}{2}S_i\right)^2$

where

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

McWilliams

球面上の Strain

$$\nabla = \mathbf{e}_\theta \frac{\partial}{R\partial\theta} + \mathbf{e}_\varphi \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\mathbf{u} = u_\theta \mathbf{e}_\theta + u_\varphi \mathbf{e}_\varphi$$

内積ではない

$$\nabla\mathbf{u} = \mathbf{e}_\theta\mathbf{e}_\theta \frac{1}{R}\frac{\partial u_\theta}{\partial\theta} + \mathbf{e}_\theta\mathbf{e}_\varphi \frac{1}{R}\frac{\partial u_\varphi}{\partial\theta}$$
$$+ \mathbf{e}_\varphi\mathbf{e}_\theta \frac{1}{R\sin\theta}\left[\frac{\partial u_\theta}{\partial\varphi} - \cos\theta u_\varphi\right]$$
$$+ \mathbf{e}_\varphi\mathbf{e}_\varphi \frac{1}{R\sin\theta}\left[\frac{\partial u_\varphi}{\partial\varphi} + \cos\theta u_\theta\right]$$
$$+ \mathbf{e}_\theta\mathbf{e}_R\left[-\frac{u_\theta}{R}\right] + \mathbf{e}_\varphi\mathbf{e}_R\left[-\frac{u_\varphi}{R}\right]$$

$\mathbf{e}_\theta$-$\mathbf{e}_\varphi$面内の成分の matrix

$$M = \begin{bmatrix} \frac{1}{R}\frac{\partial u_\theta}{\partial\theta}, & \frac{1}{R}\frac{\partial u_\varphi}{\partial\theta} \\ \frac{1}{R\sin\theta}\left[\frac{\partial u_\theta}{\partial\varphi} - \cos\theta u_\varphi\right], & \frac{1}{R\sin\theta}\left[\frac{\partial u_\varphi}{\partial\varphi} + \cos\theta u_\theta\right] \end{bmatrix}$$

非対角成分の差 ($\mathbf{e}_\theta\mathbf{e}_\varphi$成分 $-$ $\mathbf{e}_\varphi\mathbf{e}_\theta$成分)

$$\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\varphi) - \frac{\partial u_\theta}{\partial\varphi}\right] = \omega_R$$

対角成分の和

$$\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{\partial u_\varphi}{\partial\varphi}\right] = \nabla\cdot\mathbf{u}$$

平面の場合と同様に固有値の虚実で分類

$$\mathcal{D} = -4\det[M]$$

where

$$M = \frac{1}{R^2\sin^2\theta}\begin{bmatrix} \sin\theta\dfrac{\partial^2\psi}{\partial\theta\partial\varphi} - \cos\theta\dfrac{\partial\psi}{\partial\varphi}, & -\sin^2\theta\dfrac{\partial^2\psi}{\partial\theta^2} \\ \dfrac{\partial^2\psi}{\partial\varphi^2} + \sin\theta\cos\theta\dfrac{\partial\psi}{\partial\theta}, & -\left(\sin\theta\dfrac{\partial^2\psi}{\partial\theta\partial\varphi} - \cos\theta\dfrac{\partial\psi}{\partial\varphi}\right) \end{bmatrix}$$

$$= \frac{1}{R^2}\begin{bmatrix} -\left(\dfrac{\partial^2\psi}{\partial\mu\partial\varphi} + \dfrac{\mu}{1-\mu^2}\dfrac{\partial\psi}{\partial\varphi}\right), & -(1-\mu^2)\dfrac{\partial^2\psi}{\partial\mu^2} + \mu\dfrac{\partial\psi}{\partial\mu} \\ \dfrac{1}{1-\mu^2}\dfrac{\partial^2\psi}{\partial\varphi^2} - \mu\dfrac{\partial\psi}{\partial\mu}, & \dfrac{\partial^2\psi}{\partial\mu\partial\varphi} + \dfrac{\mu}{1-\mu^2}\dfrac{\partial\psi}{\partial\varphi} \end{bmatrix}$$

$$\therefore \mathcal{D} = \frac{4}{R^4}\left[\left(\frac{\partial^2\psi}{\partial\mu\partial\varphi} + \frac{\mu}{1-\mu^2}\frac{\partial\psi}{\partial\varphi}\right)^2 - \left(\frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\varphi^2} - \mu\frac{\partial\psi}{\partial\mu}\right)\left((1-\mu^2)\frac{\partial^2\psi}{\partial\mu^2} - \mu\frac{\partial\psi}{\partial\mu}\right)\right]$$

$$\begin{cases} \mathcal{D} > 0 \rightarrow \text{伸縮部分} \\ \mathcal{D} < 0 \rightarrow \text{渦部分} \end{cases}$$