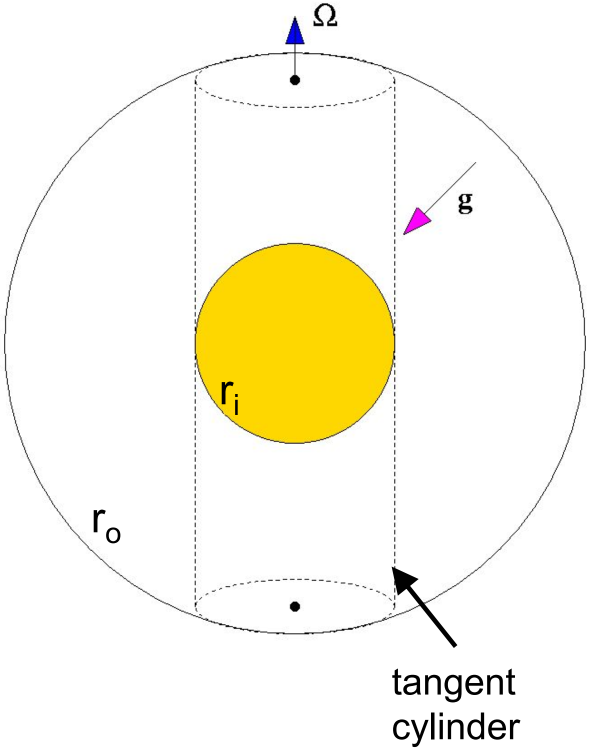
Convection in rotating spherical shells

Consider a rotating spherical shell with inner radius r_i and outer radius r_o. The ratio r_i/r_o is 0.35 for the Earth's core, ≈0.85 for Jupiter's and ≈0.5 for Saturn's molecular H_2 layer.

We use spherical coordinates (r,θ,φ) and sometimes cylindrical coordinates (z,s,φ). In contrast to the plane layer case, the direction between the radial gravity vector $\mathbf{g}=-g\mathbf{e}_r$ and the cylindrical rotation $\mathbf{\Omega}=\Omega\mathbf{e}_z$ differs in different parts of the shell: near the poles the two vectors are nearly parallel and in the equatorial plane they are perpendicular.

The inner core **tangent cylinder** separates three dynamically distinct regions of the shell. Because of the rigidity of the flow in z-direction imposed by the Proudman-Taylor constraint, consider columns of fluid aligned with the z-direction. Columns outside the tangent cylinder end at two points on the outer boundary, those inside the tangent cylinder region stretch between outer and inner boundary.



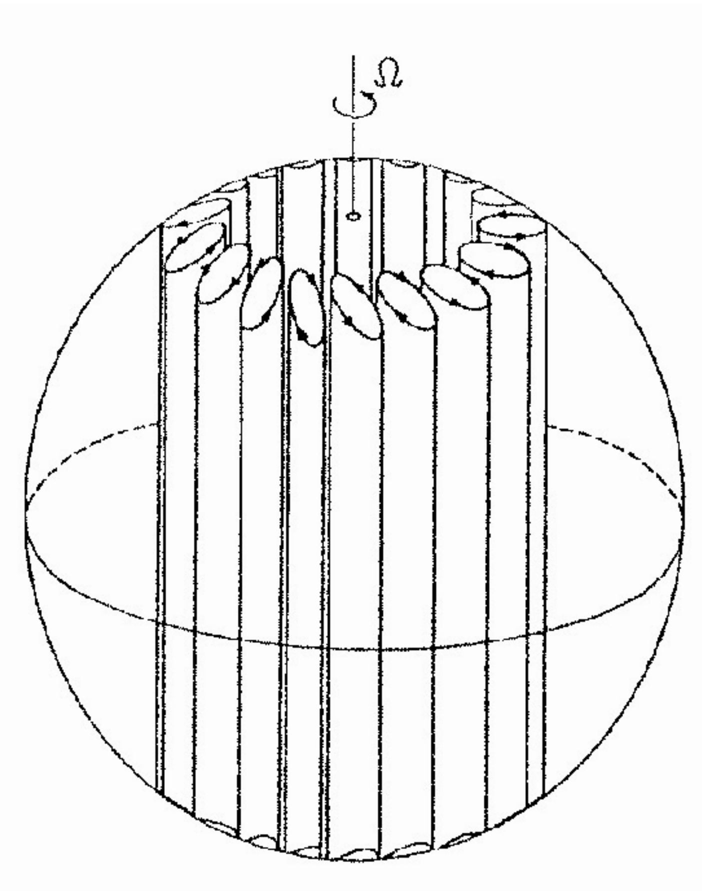
For a fluid shell heated from below (e.g. by an imposed ΔT between inner and outer boundary) or from within, we define the Rayleigh number and the Ekman number as before, using the shell thickness $D=r_o-r_i$ as the basic length scale.

Symbols: r_i – inner shell radius, r_o – outer radius, $D=r_o-r_i$ - shell thickness

Onset of convection in a sphere

As in cartesian geometry, incompressible flow in a sphere can be represented by poloidal and toroidal components (where r instead of z relevant direction, e.g. the toroidal flow has no r-component). In the case $E \to \infty$ the solution to the linearized problem can be expanded in spherical harmonic functions in the θ,φ-coordinates (which are eigenfunctions of the horizontal part of the Laplacian operator). For finite E, a separation of variables (r,θ) is not possible and even the solution of the linear stability problem requires numerical techniques. We will qualitatively consider the solution of the marginal stability problem and rationalize it with what we learned from the cartesian problem. We consider again the relevant case $E \ll 1$.

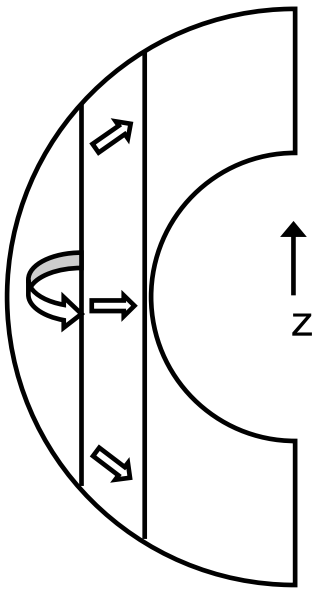
Convection starts in the form of columns outside the tangent cylinder parallel to the rotation axis. The basic flow is a vortex motion around the axis of the column, There is some analogy to the cartesian case with $\mathbf{g} \perp \mathbf{\Omega}$: the flow is such as to satisfies the P-T-theorem as far as possible.



The solution must be periodic in φ and can be written as $f(r,\theta)\exp(i[m\varphi-\omega t])$.

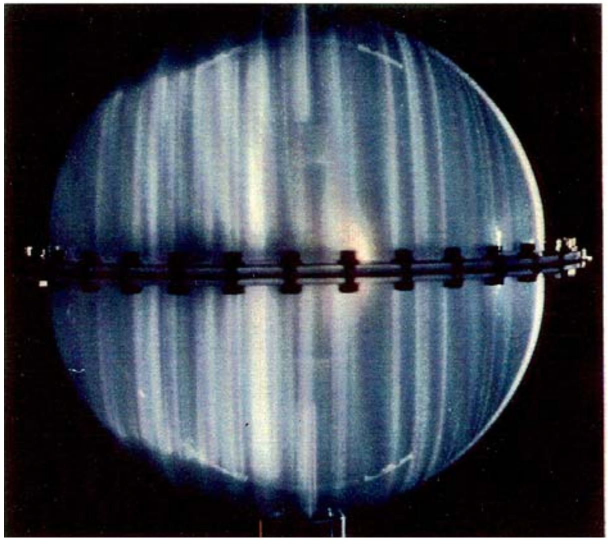
Onset of convection in a sphere II

The Proudman-Taylor theorem cannot be satisfied precisely, because the velocity in the columns must adapt to the sloping boundaries at both ends. This involves for example a velocity component u_z that must change with z. Because the slope becomes larger with distance from the rotation axis, the columns tend to cluster around the inner core as close as possible to the rotation axis, i.e. near the tangent cylinder boundary.



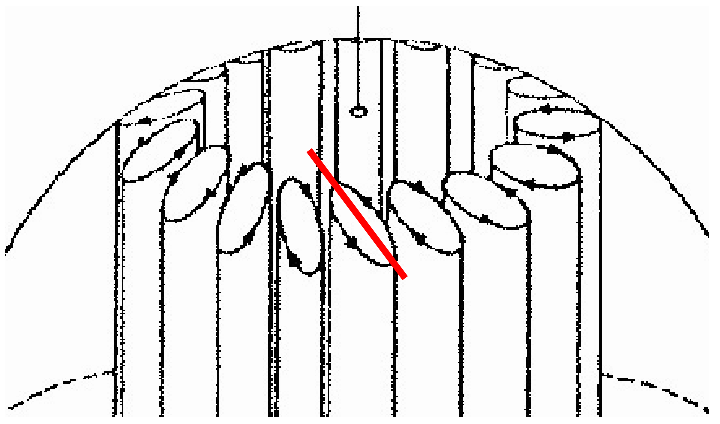
Because the Proudman-Taylor theorem must be violated, which requires that viscous friction balances the Coriolis force and the flow length scale must become small to achieve this effect, the same strong dependence on the Ekman number applies as in the plane layer case:

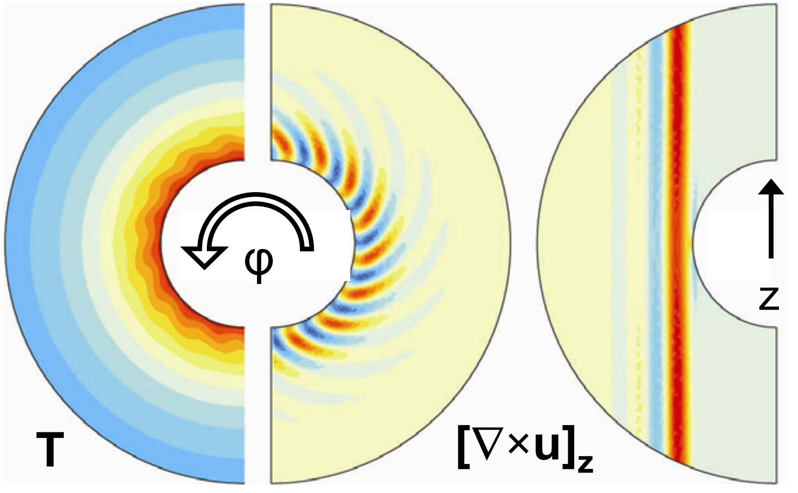
$$Ra_{crit} \sim E^{-4/3} \qquad m_{crit} \sim E^{-1/3}$$



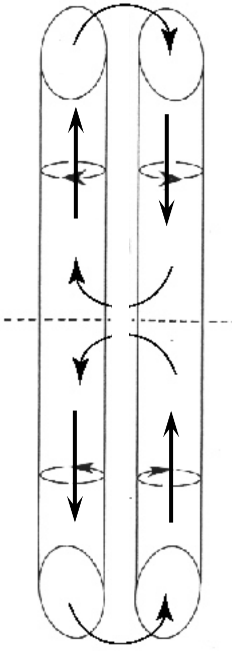
At very low E, the number of columns is very large, e.g. $m=10^5$ for $E=10^{-15}$. They become thinner than spaghetti. $\omega \neq 0$ for finite values of Pr $\Rightarrow$ the columns drift in longitude at constant rate.

Convection in a rotating sphere





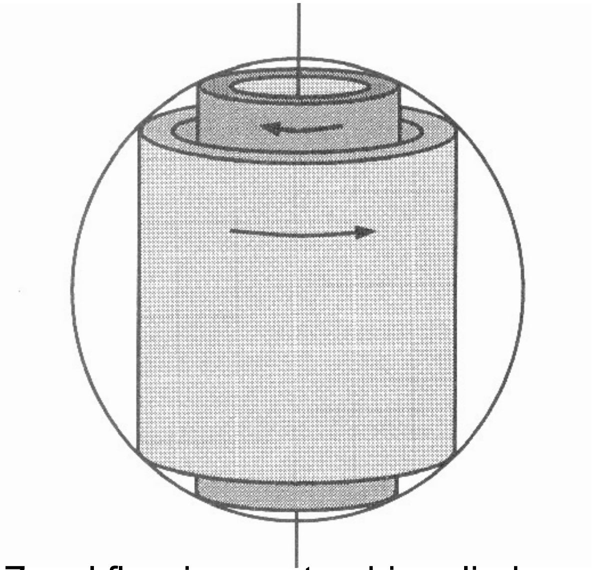
Numerical calculation at $E=10^{-5}$, Pr=1, Ra=1.5 Ra_{crit}



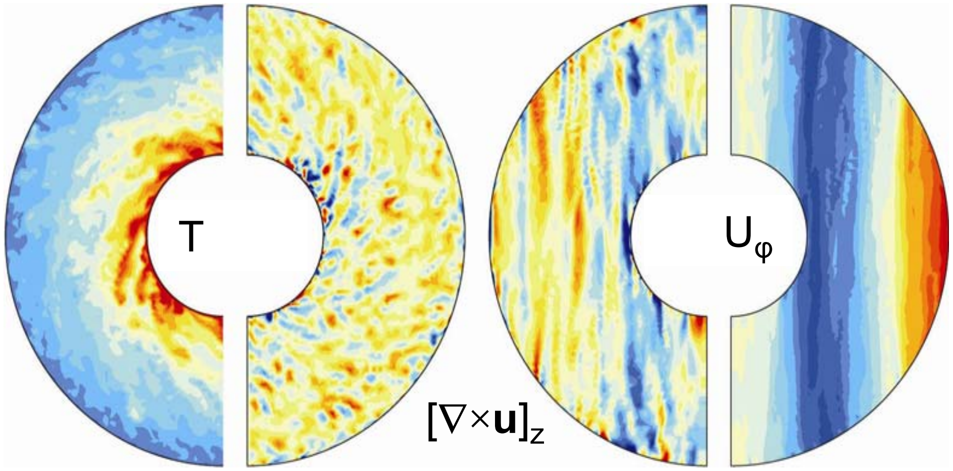
Consider cross sections through the columns ⇒ they are elongated and their long axis is tilted with respect to the s-direction. The cause for the tilt is the curvature of the boundary at which the column ends.

In addition to vortex motion, there is flow along the column axis, diverging away from the equatorial plane in anticyclonic columns and converging in cyclonic ones. At higher Ra this flow can become comparable in magnitude with the vortex motion. It makes the flow helical, with negative helocity in the northern hemisphere and positive in the southern. Effects that give rise to this motion are (1) Ekman pumping (for rigid boundaries), (2) concentration of temperature anomaly in equatorial plane, (3) boundary curvature.

Zonal flow excited by convection



Zonal flow in geostrophic cylinders



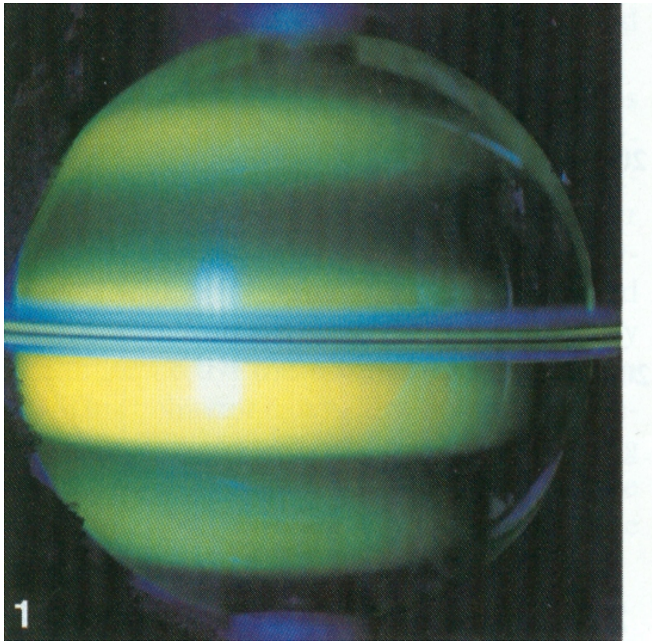
Numerical calculation at $E=10^{-5}$, $Pr=1$, $Ra = 45\,Ra_{crit}$

At low Ekman number and high Rayleigh number

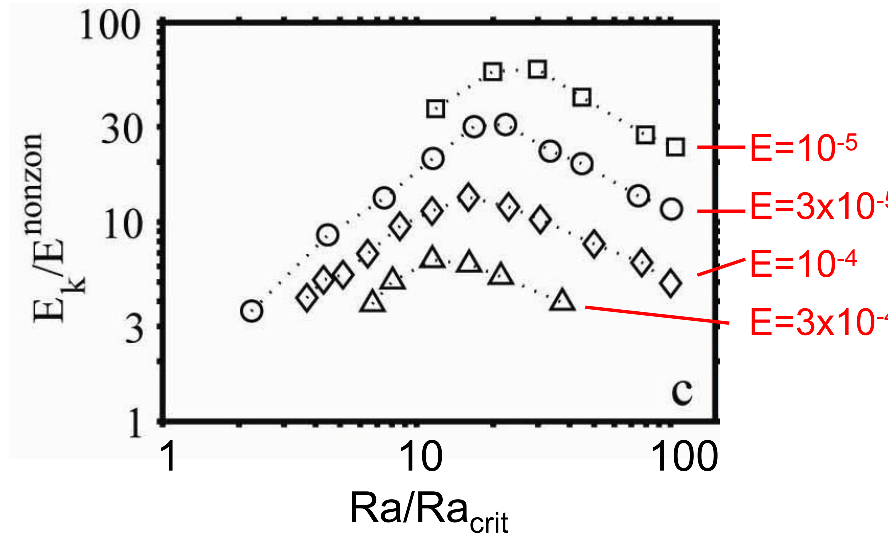
- convection fills the entire spherical shell including the tangent cylinder
- the perfect columnar structure is broken up, but preferred orientation parallel to z
- a strong zonal flow (i.e. a mean flow in φ-direction) $U_\varphi(s,z) = \langle u_\varphi \rangle_\varphi$ is excited
- the zonal flow is geostrophic (independent of z $\Rightarrow$ no violation of P-T--theorem) and prograde (eastward) at large s and retrograde (westward) at smaller s
- with free-slip boundaries the zonal flow can be much stronger than convection

Symbols (notation: $\langle\ \rangle_\varphi$ - average over φ) U_φ – mean flow in φ-direction

Zonal flow excited by convection II

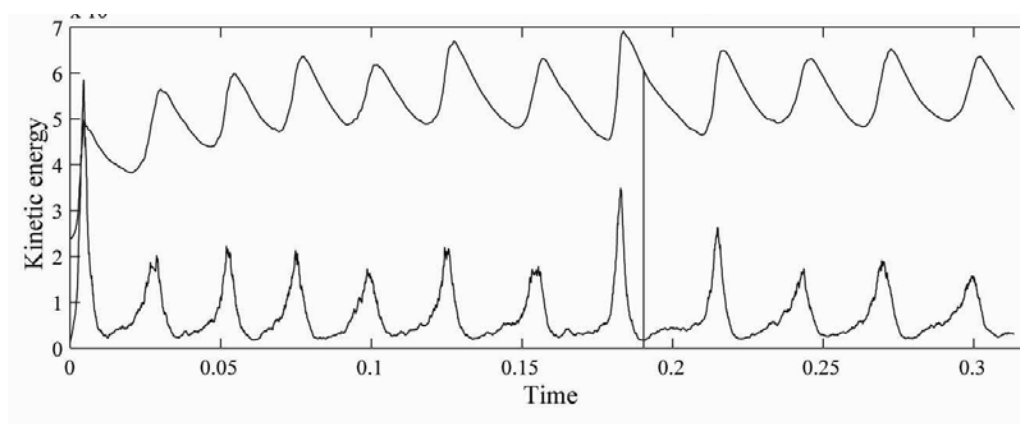


Laboratory convection rotating sphere

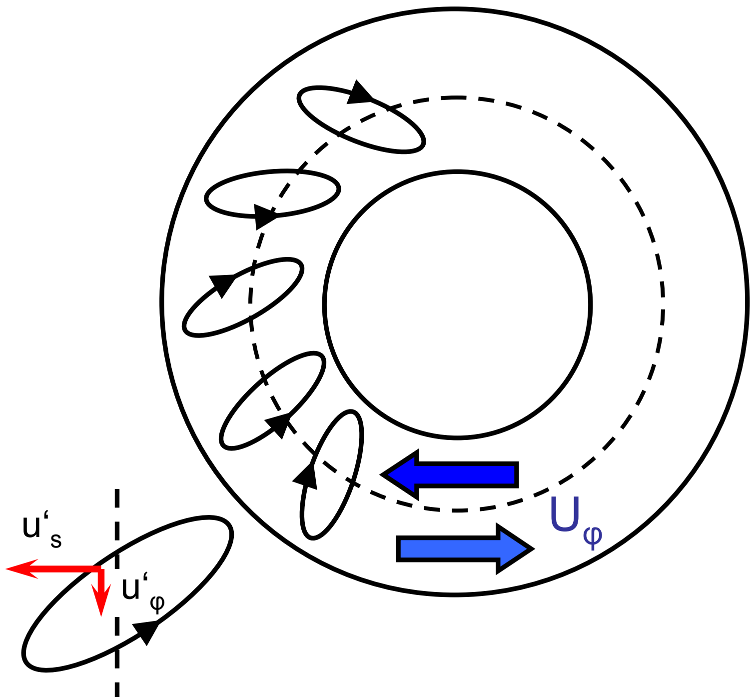


Ratio of total kinetic energy to the energy of the non-zonal (convective) part of the flow in numerical simulations of convection in a spherical shell with free-slip boundaries

Relaxation oscillations at low Ekman number and moderate Ra ≈ 5 – 10 Ra_{crit} Short pulses of vigorous convection transfer energy into the zonal flow. The shear of the zonal flow suppresses convection and the zonal flow loses energy until the next convective burst can occur.



Reynolds stress and zonal flow



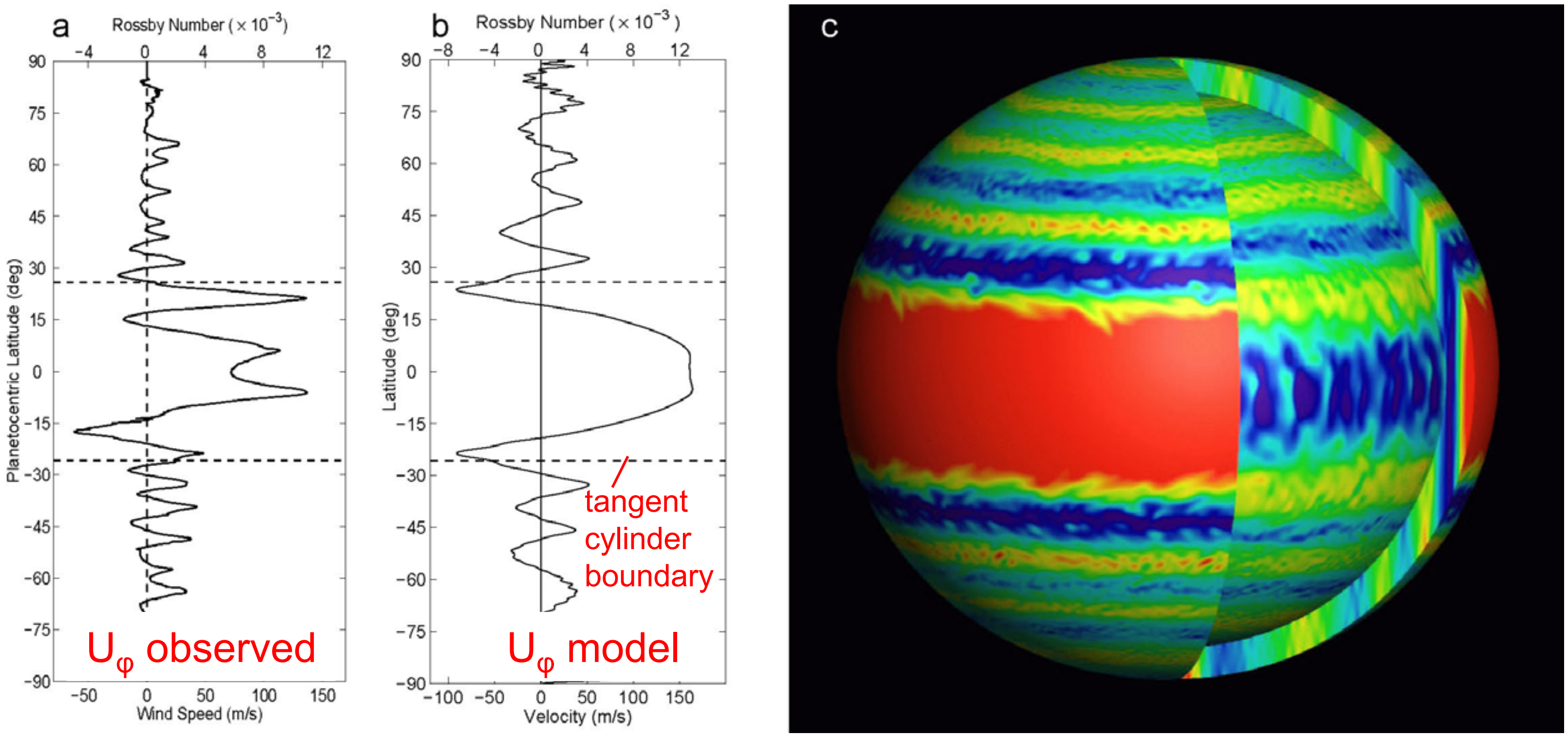
For simplicity, assume two dimensional flow **u**(s,z). Separate into large-scale (here: zonal) and small-scale part: $\mathbf{u} = U_\varphi(s)\mathbf{e}_\varphi + \mathbf{u}'(s,\varphi)$ Insert into Navier-Stokes equation and take average in φ-direction of its φ-component (note that $\langle \mathbf{u}' \rangle_\varphi = 0$):

$$\partial U_\varphi / \partial t + \partial \langle u'_s u'_\varphi \rangle / \partial s = E \nabla^2 U_\varphi$$

The term involving $t_{s\varphi} = \langle u'_s u'_\varphi \rangle_\varphi$ arises from the $\mathbf{u}\cdot\nabla\mathbf{u}$ -term in the N-S-equation. Formally $t_{s\varphi}$ can be considered as a component of an external stress tensor acting on the large-scale flow.

Similar considerations can be made for general (3D) flows. The resulting stress tensor is called **Reynolds stress.** In turbulent boundary layers, the Reynolds stress adds to the retarding viscous stress and transfers energy from the large-scale flow to small scales (**turbulent cascade**), where it is destroyed by friction. In our case, the systematic tilt of the convective elements leads to positive correlation of u'_s and u'_φ. The Reynolds stress transports (angular) momentum in +s-direction, adding to the zonal flow. It transfers energy from the small scales to the large-scale flow (**inverse cascade**). The Reynolds stress does not need to be very large, but since only the weak (at low E) large-scale viscous stress counteracts the (fully geostrophic) zonal flow, it can accumulate much energy with time.

Application to the gas planets



Convection model for Jupiter (Heimpel et al., 2005) with a thin convecting shell (r_i/r_o=0.85). Colors in c) show u_φ. On the surface U_φ has the right magnitude with strong prograde flow near the equator and alternating bands of weaker zonal at higher latitude (inside the tangent cylinder). The Boussinesq model does not explain the observed asymmetries and ignores the strong density variations with radius. Nonetheless it supports the hypothesis of an internal origin of the zonal jet flow at the surface of the gas planets.