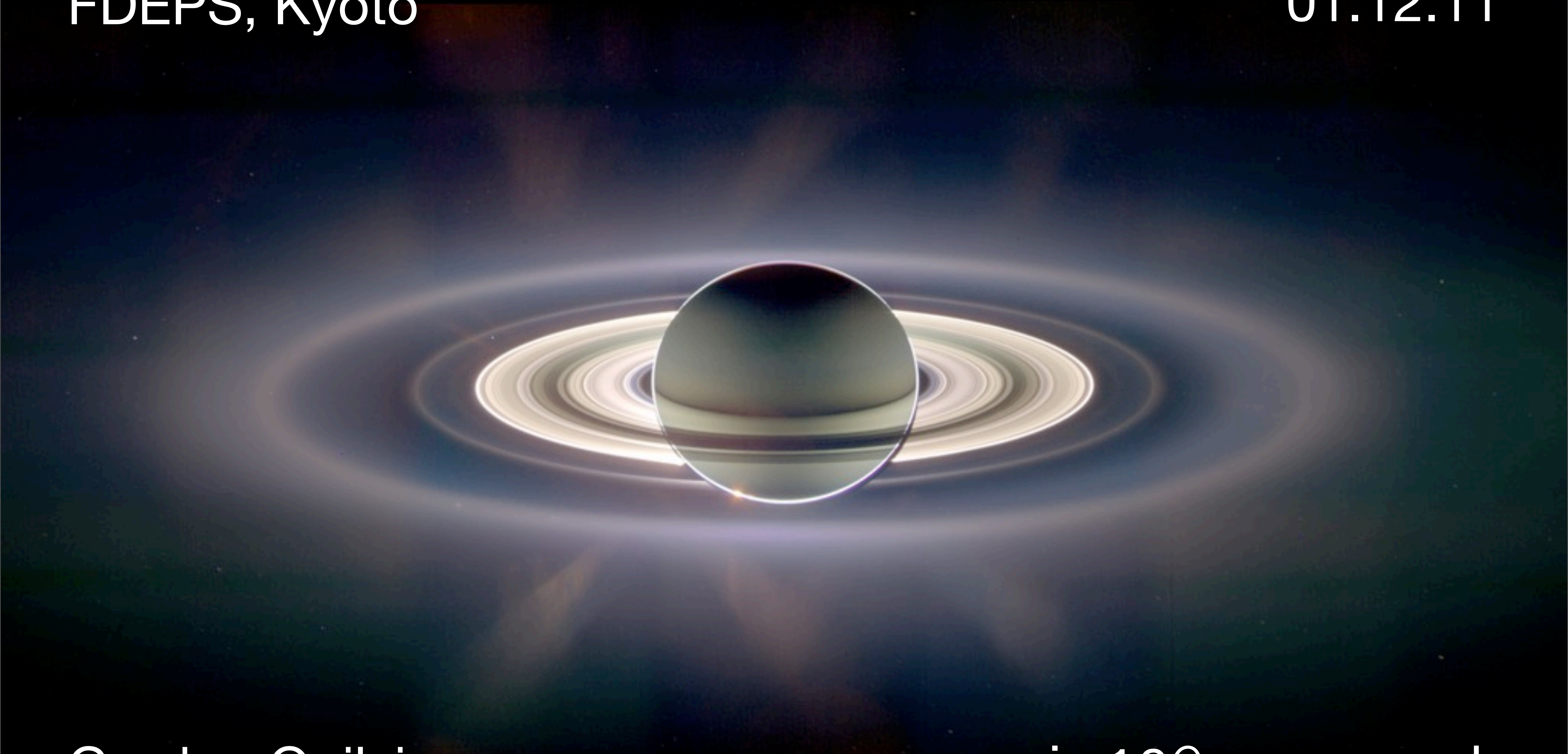
Dynamics of astrophysical discs

Lecture 4

FDEPS, Kyoto

01.12.11



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2D compressible dynamics in shearing sheet

- 2D compressible sheet: inviscid, self-gravitating
- Surface density $\Sigma(x,y,t)$
- 2D pressure $P(x,y,t)$
  - Relate to vertically integrated quantities $\int \rho\,\mathrm{d}z,\ \int p\,\mathrm{d}z$

    but only a model, not derivable exactly from 3D equations
- Basic equations:

$$\frac{\partial\Sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\Sigma\boldsymbol{u}) = 0$$

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\boldsymbol{\nabla}\overset{\substack{\Phi = -\Omega S x^2 \\ \big\downarrow}}{\Phi} - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}P$$

- Disc potential $\Phi_{\mathrm{d}}(x,y,z,t)$ satisfies $\nabla^2\Phi_{\mathrm{d}} = 4\pi G\Sigma\,\delta(z)$
- Then evaluate in midplane: $\Phi_{\mathrm{d,m}}(x,y,t) = \Phi_{\mathrm{d}}(x,y,0,t)$
- Assume barotropic relation $P = P(\Sigma)$ for simplicity

2D compressible dynamics in shearing sheet

- Solve Poisson's equation in Fourier domain:

$$\tilde{\Sigma}(k_x, k_y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Sigma(x, y, t)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{etc.}$$

$$\nabla^2 \Phi_\mathrm{d} = 4\pi G \Sigma\, \delta(z)$$

$$\Rightarrow \left(-k^2 + \frac{\partial^2}{\partial z^2}\right) \tilde{\Phi}_\mathrm{d} = 4\pi G \tilde{\Sigma}\, \delta(z) \qquad k = (k_x^2 + k_y^2)^{1/2}$$

$$\Rightarrow \tilde{\Phi}_\mathrm{d} = -\frac{2\pi G \tilde{\Sigma}}{k}\, \mathrm{e}^{-k|z|} \quad (k \neq 0) \qquad \text{so that} \left[\frac{\partial \tilde{\Phi}_\mathrm{d}}{\partial z}\right]_{0-}^{0+} = 4\pi G \tilde{\Sigma}$$

$$\Rightarrow \tilde{\Phi}_\mathrm{d,m} = -\frac{2\pi G \tilde{\Sigma}}{k}$$

- $k = 0$ component gives no horizontal force anyway

2D compressible dynamics in shearing sheet

- Conservation of potential vorticity / "vortensity":

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla} P$$

- Use identity $(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{u}|^2)$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + [(2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u}] = \boldsymbol{\nabla}(\cdots)$$

since $P = P(\Sigma)$

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{u}) + \boldsymbol{\nabla} \times [(2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u}] = \boldsymbol{0}$$

- Use identity $\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla} \boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A})$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)(2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \boldsymbol{u}) = (2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \boldsymbol{u})(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

since 2D

$$= -(2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \boldsymbol{u})\frac{1}{\Sigma}\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\Sigma$$

$$\Rightarrow \quad \frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

where $q = \dfrac{2\Omega + (\boldsymbol{\nabla} \times \boldsymbol{u})_z}{\Sigma}$

2D compressible dynamics in shearing sheet

- Conservation of potential vorticity / “vortensity”:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \text{where} \quad q = \frac{2\Omega + (\boldsymbol{\nabla} \times \boldsymbol{u})_z}{\Sigma}$$

- Recall $\boldsymbol{u} = -Sx\,\boldsymbol{e}_y + \boldsymbol{v}$:

$$\left(\frac{\partial}{\partial t} - Sx\,\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) q = 0 \qquad\qquad q = \frac{2\Omega - S + (\boldsymbol{\nabla} \times \boldsymbol{v})_z}{\Sigma}$$

- Unlike incompressible 2D case, vortex dynamics not the whole story
- Vortical disturbances are coupled to acoustic ones

2D compressible dynamics in shearing sheet

- Linear stability of uniform 2D self-gravitating sheet

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{u}) = 0$$

$$\Phi = -\Omega S x^2$$

$$\downarrow$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}P$$

$$\nabla^2 \Phi_{\mathrm{d}} = 4\pi G \Sigma \, \delta(z)$$

- Basic state: $\Sigma = \mathrm{cst}, \quad \boldsymbol{u} = -Sx\,\boldsymbol{e}_y$
- Linearized equations for perturbations Σ', $\boldsymbol{v}$, etc.:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\Sigma' + \Sigma\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$$

$$P' = v_{\mathrm{s}}^2 \Sigma'$$

$$\downarrow$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\boldsymbol{v} - Sv_x\,\boldsymbol{e}_y + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\boldsymbol{\nabla}\Phi'_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}P'$$

$$\nabla^2 \Phi'_{\mathrm{d}} = 4\pi G \Sigma' \, \delta(z)$$

sound speed v_{s}

2D compressible dynamics in shearing sheet

- Solutions are shearing waves:

$$\Sigma'(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\Sigma}'(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\} \qquad \text{etc.}$$

- Amplitude equations:

$$\frac{\mathrm{d}\tilde{\Sigma}'}{\mathrm{d}t} + \Sigma\,\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$$

$$\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y = -\mathrm{i}k_x\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right)$$

$$\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x = -\mathrm{i}k_y\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right)$$

$$\tilde{\Phi}'_{\mathrm{d,m}} = -\frac{2\pi G\tilde{\Sigma}'}{k}$$

- Vortensity perturbation $\tilde{q}' = \dfrac{\mathrm{i}k_x\tilde{v}_y - \mathrm{i}k_y\tilde{v}_x}{\Sigma} - \dfrac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2}$

  satisfies $\dfrac{\mathrm{d}\tilde{q}'}{\mathrm{d}t} = 0$ as expected [exercise]

2D compressible dynamics in shearing sheet

- Consider axisymmetric waves: $k_y = 0,\ k_x = \mathrm{cst},\ k = |k_x|$
- Amplitudes $\propto \mathrm{e}^{-\mathrm{i}\omega t}$

$$-\mathrm{i}\omega\tilde{\Sigma}' + \Sigma\,\mathrm{i}k_x\tilde{v}_x = 0$$

$$-\mathrm{i}\omega\tilde{v}_x - 2\Omega\tilde{v}_y = -\mathrm{i}k_x\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\frac{\tilde{\Sigma}'}{\Sigma}$$

$$-\mathrm{i}\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x = 0$$

- Multiply second equation by $\mathrm{i}\omega$ and eliminate $\tilde{\Sigma}'$ and $\tilde{v}_y$:

$$\omega^2\tilde{v}_x - 2\Omega(2\Omega - S)\tilde{v}_x = k_x^2\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\tilde{v}_x$$

- Deduce dispersion relation for “density waves”:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + v_\mathrm{s}^2k_x^2$$

- Also vortical solution $\omega = 0,\ \tilde{v}_x = 0$: zonal flow / geostrophic flow

2D compressible dynamics in shearing sheet

- Dispersion relation for density waves:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + v_\mathrm{s}^2 k_x^2$$

inertial acoustic (restoring forces)

self-gravity (destabilizing)

- "Acoustic-inertial waves"
- Disc is unstable to axisymmetric disturbances if $\omega^2 < 0$ for some k_x
- ω^2 is minimized with respect to $|k_x|$ when

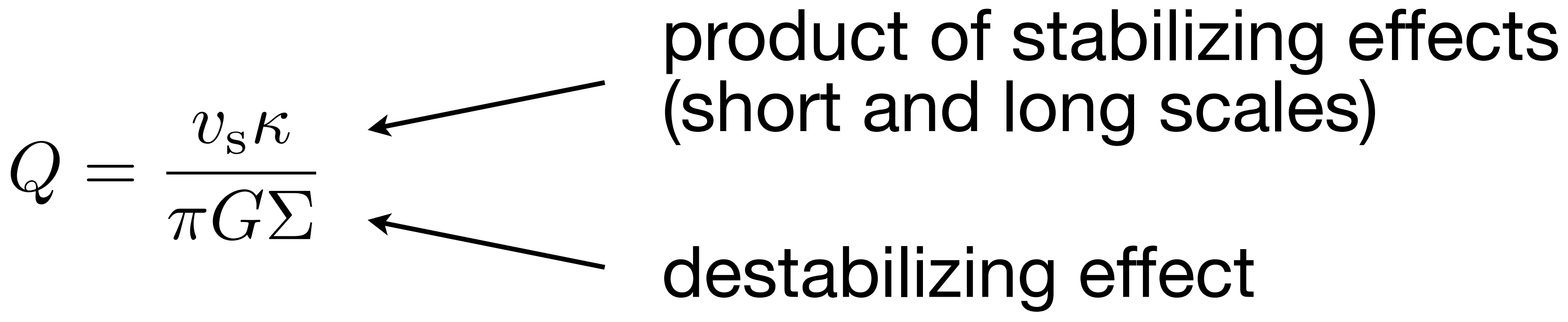
$$0 = -2\pi G\Sigma + 2v_\mathrm{s}^2|k_x| \qquad \Rightarrow \quad |k_x| = \frac{\pi G\Sigma}{v_\mathrm{s}^2}$$

$$(\omega^2)_\mathrm{min} = \kappa^2 - \frac{(\pi G\Sigma)^2}{v_\mathrm{s}^2} = \kappa^2\left(1 - \frac{1}{Q^2}\right)$$

- Gravitational instability if $Q < 1$, where $Q = \dfrac{v_\mathrm{s}\kappa}{\pi G\Sigma}$
  (Toomre stability parameter)

2D compressible dynamics in shearing sheet

- Gravitational instability if $Q < 1$, where $Q = \dfrac{v_\mathrm{s}\kappa}{\pi G \Sigma}$
- Toomre stability parameter Q :
  - An inverse measure of self-gravity
  - A measure of temperature

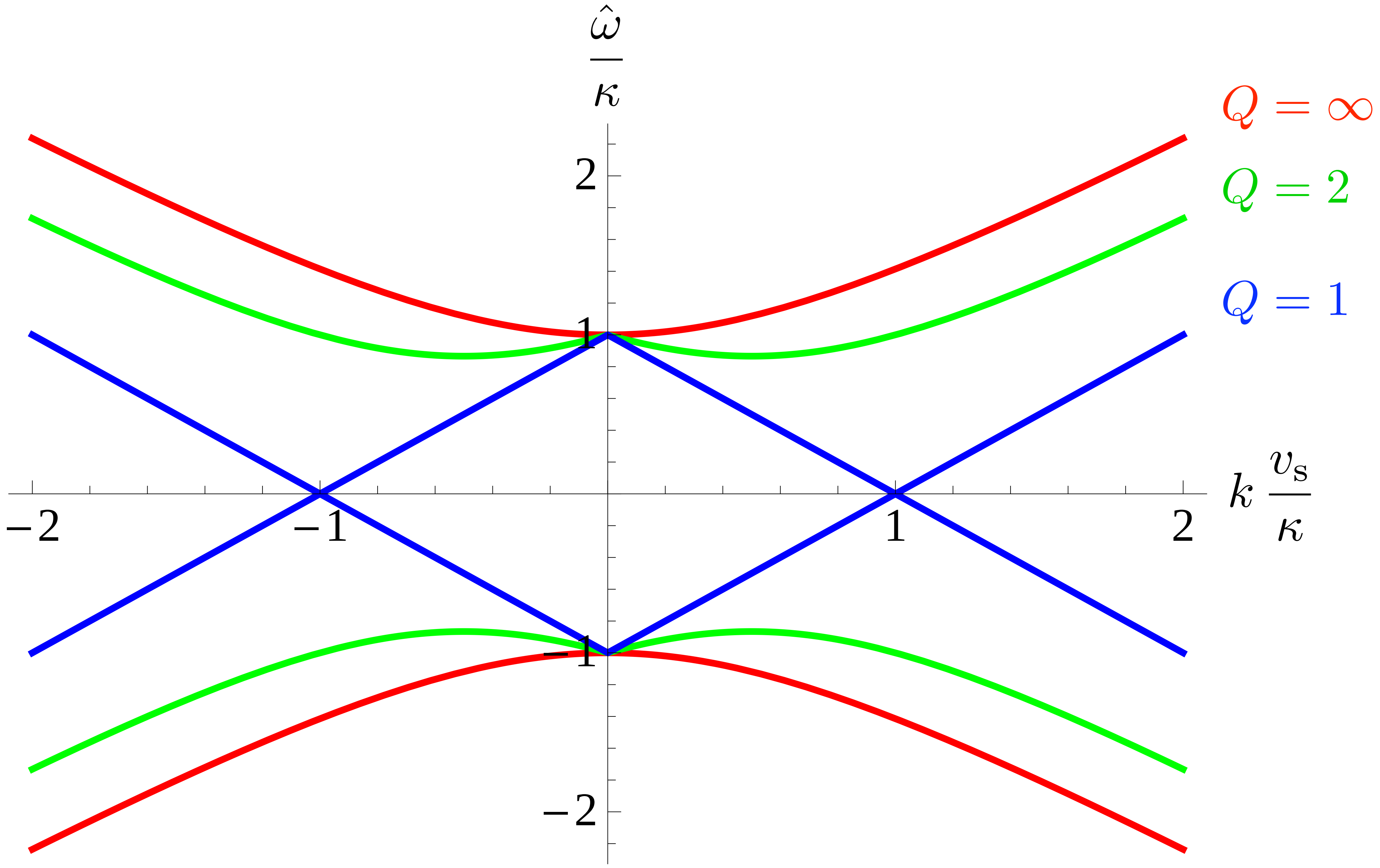


$$Q = \frac{v_\mathrm{s}\kappa}{\pi G \Sigma}$$

product of stabilizing effects
(short and long scales)

destabilizing effect

2D compressible dynamics in shearing sheet



$$Q = \frac{v_\mathrm{s}\kappa}{\pi G \Sigma}$$

2D compressible dynamics in shearing sheet

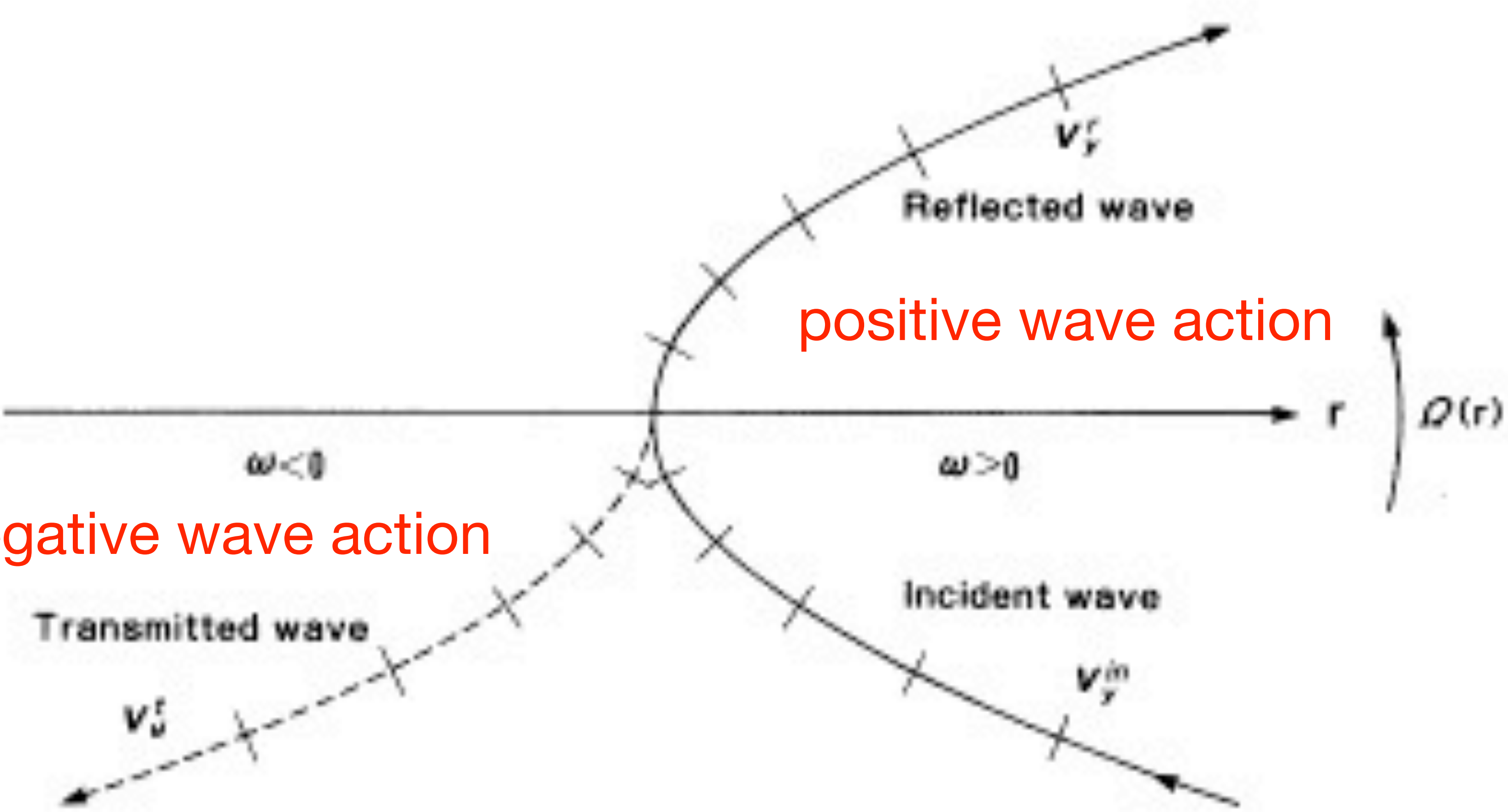
Nakagawa & Sekiya (1992)

Non-vortical perturbations

Oscillating at ends of swing

Over-reflection due to wave action conservation

$$|I|^2 = |R|^2 - |T|^2$$



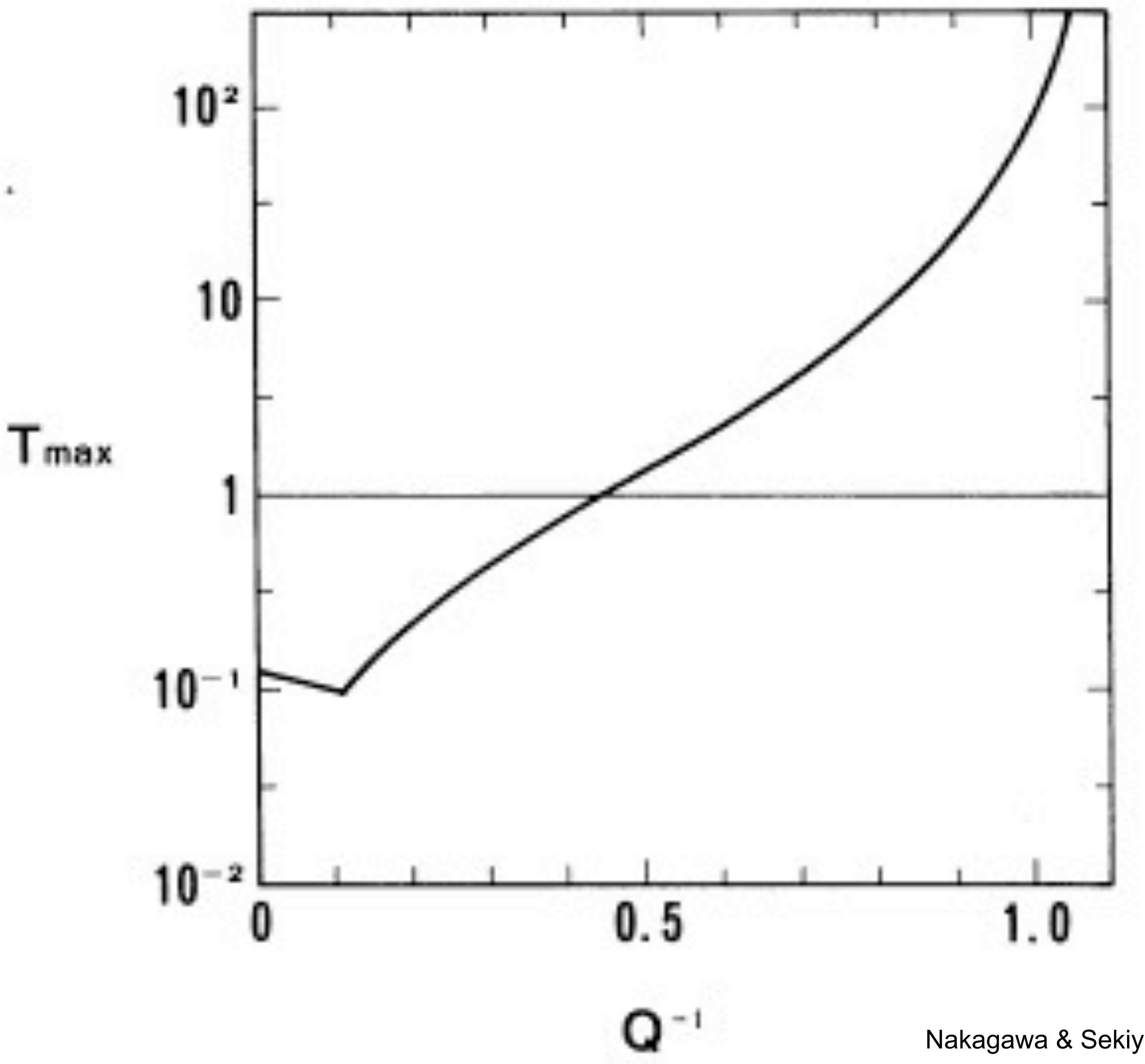
positive wave action

negative wave action

2D compressible dynamics in shearing sheet

Maximum transmission coefficient

Swing amplification due (mainly) to self-gravity



Nakagawa & Sekiya (1992)

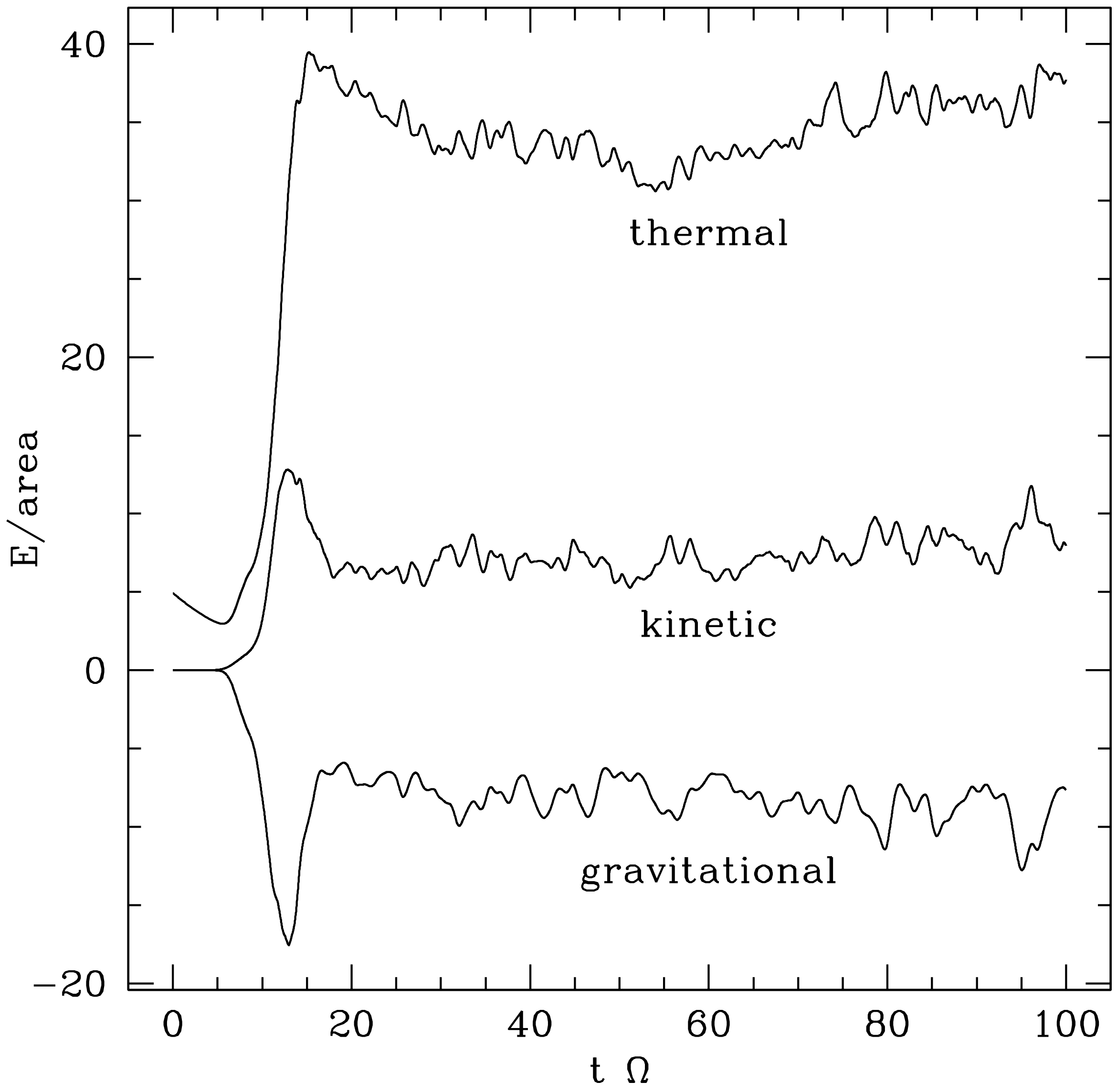
2D compressible dynamics in shearing sheet

- Occurrence of gravitational instability:
  - If $Q < 1$, disc tends to form rings (axisymmetric instability, exponential growth)
  - If $1 < Q \lesssim 1.5$, disc tends to form spiral waves or clumps (non-axisymmetric instability, transient growth)
- Since $Q \propto v_\mathrm{s} \propto T^{1/2}$, thermostatic regulation is possible: instability → motion → dissipation (shock/viscous) → heating
- Two possible outcomes of gravitational instability:
  - Fragmentation: formation of gravitationally bound objects (clumps...moonlets / planets / stars)
  - Gravitational turbulence: sustained activity of non-axisymmetric density waves (e.g. “self-gravity wakes” in Saturn’s rings)
- Efficient cooling promotes fragmentation, or enhances the efficiency of gravitational turbulence, since cooling balances viscous heating

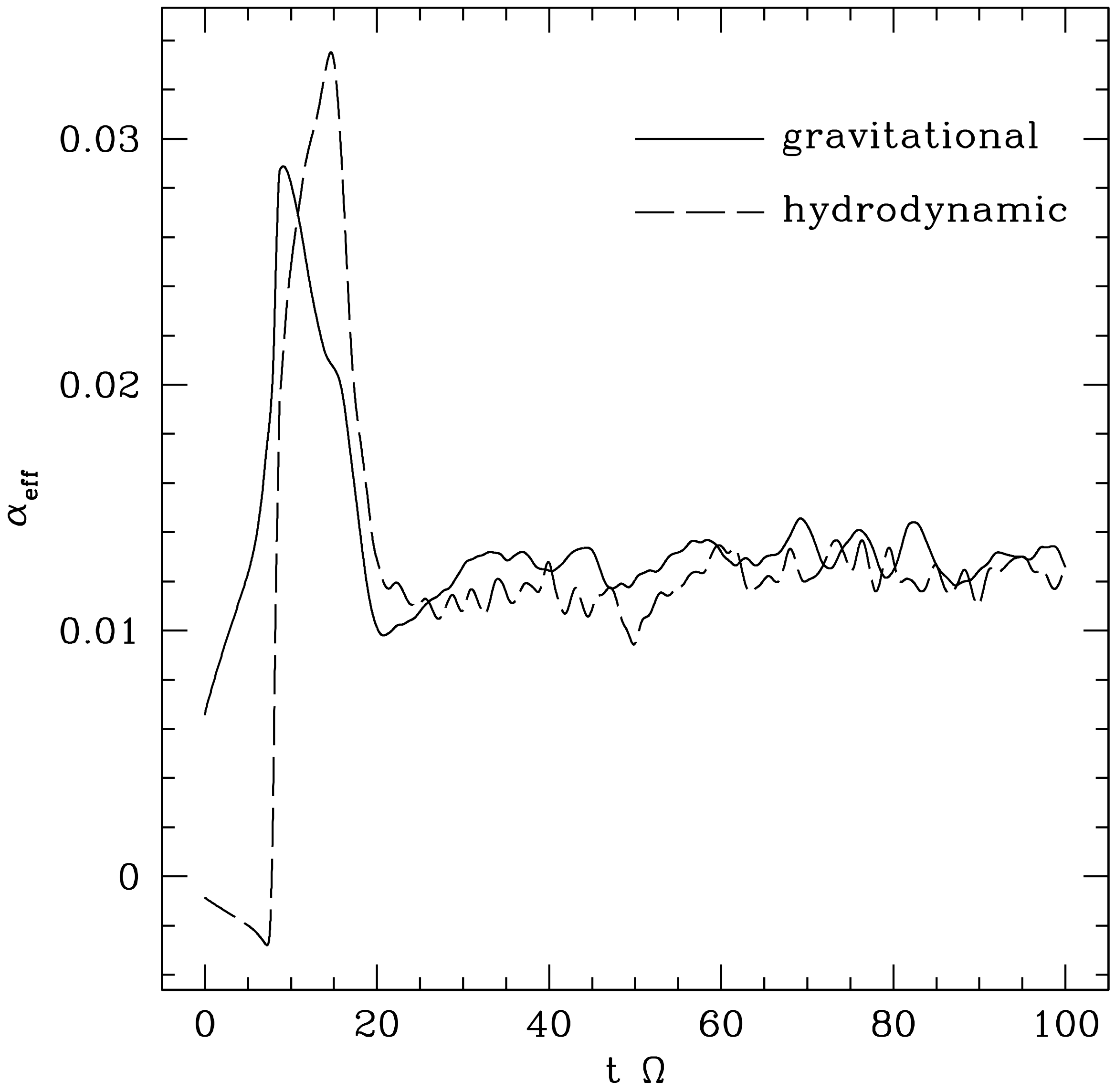
Gravitational instability

Nonlinear outcome in razor-thin disc with heating and cooling (Gammie 2001)

energy

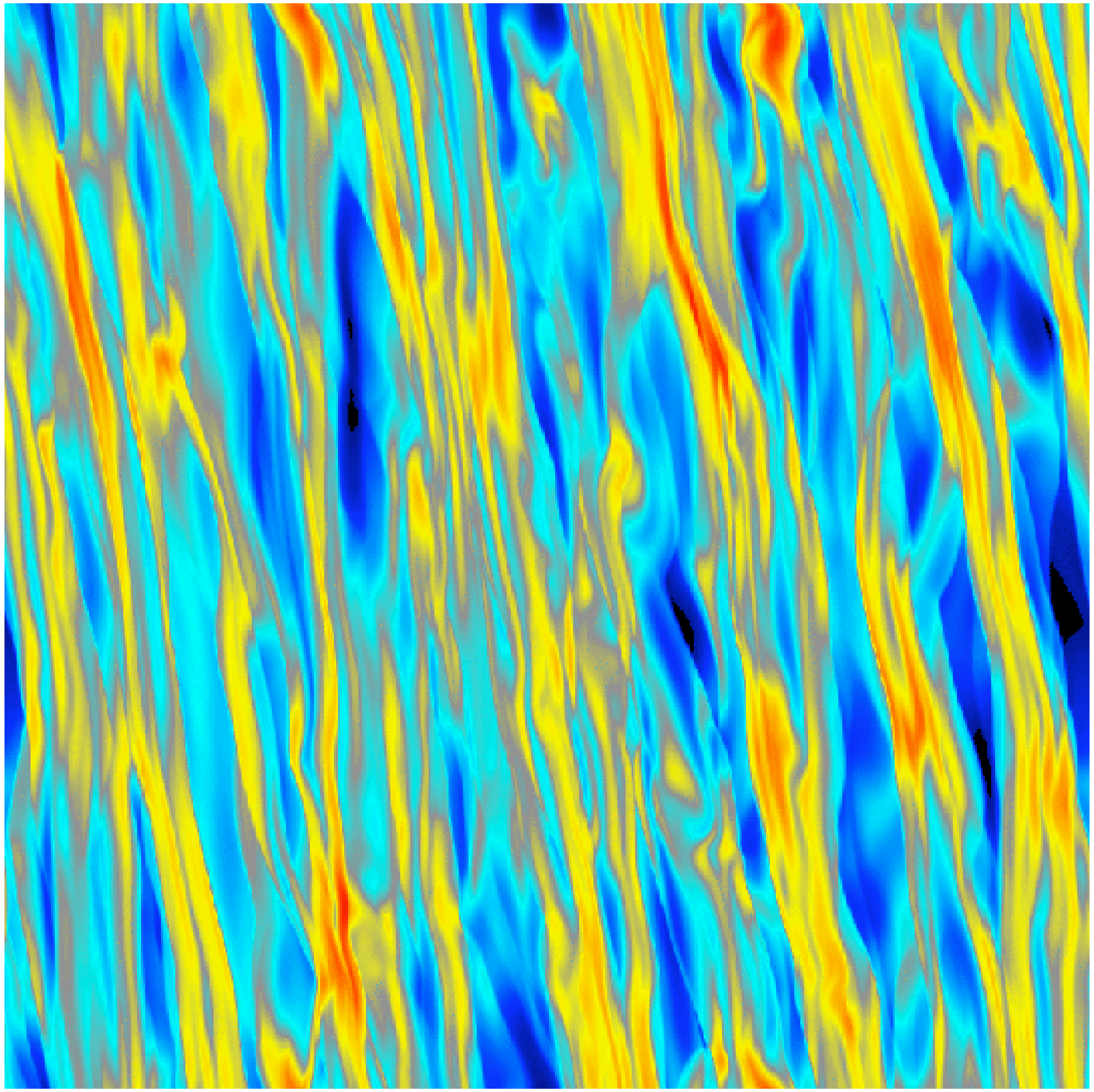


angular momentum transport



Gravitational instability

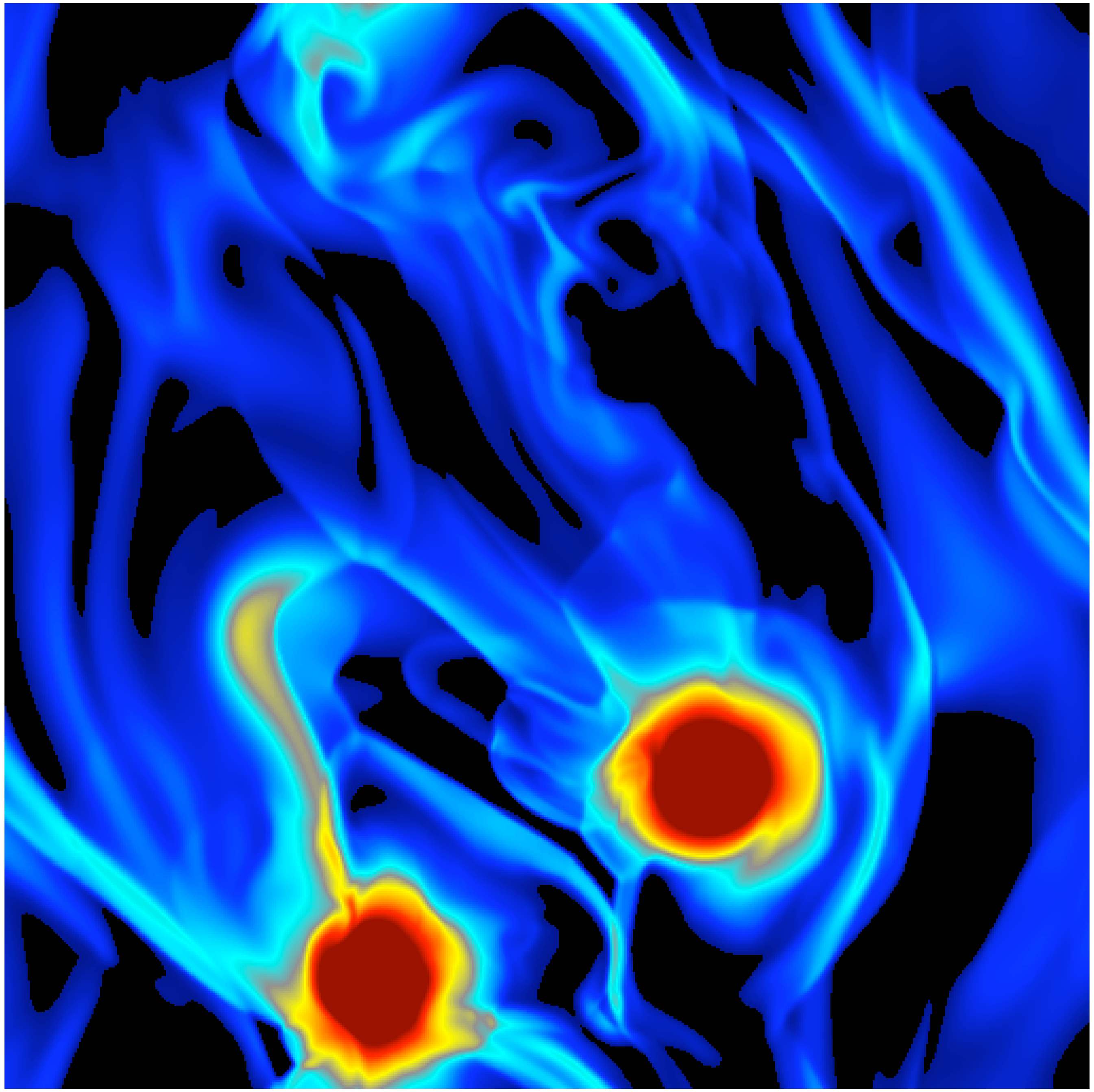
Gravitoturbulent state (slow cooling)



Gammie (2001)

Gravitational instability

Fragmentation (rapid cooling)

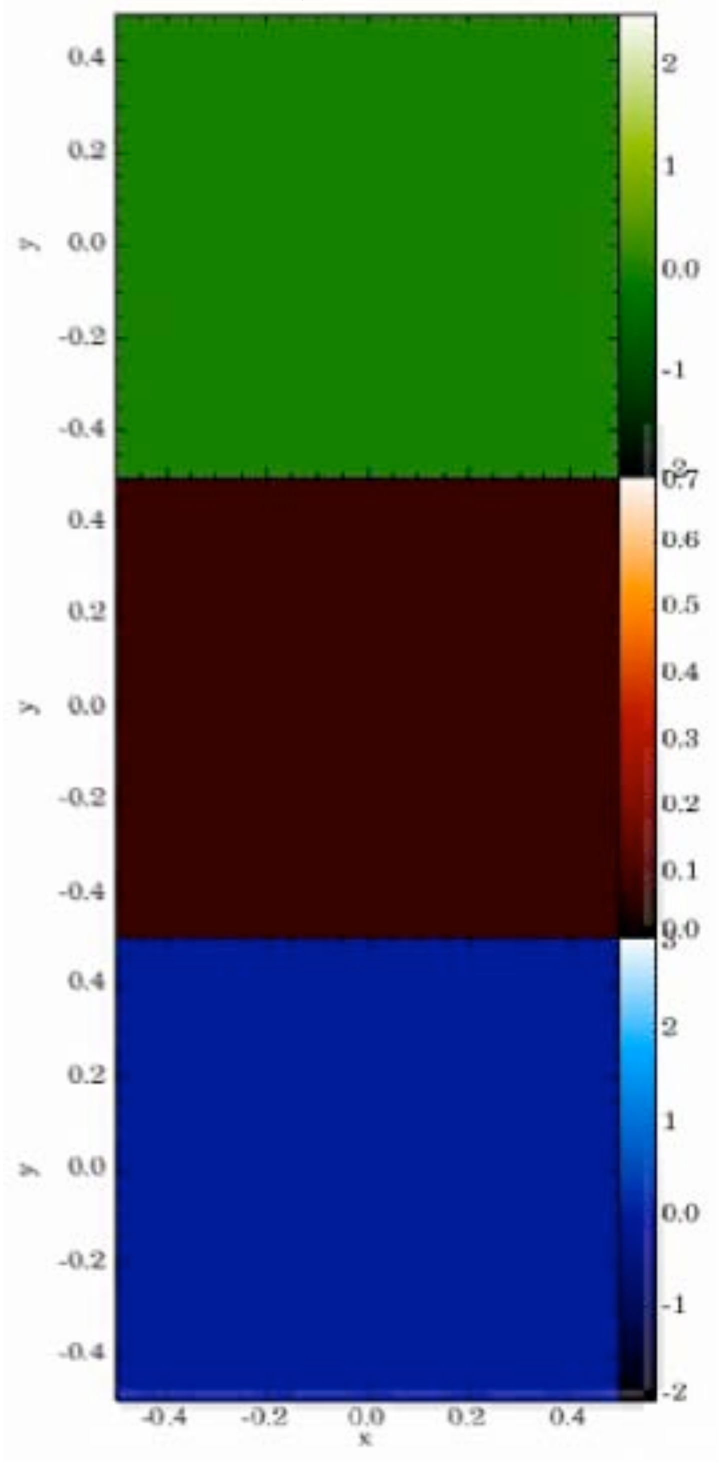


Gammie (2001)

Gravitational instability

- 2D nonlinear simulation with cooling time β/Ω (S.-J. Paardekooper)

$\beta = 4$



surface
density

Σ

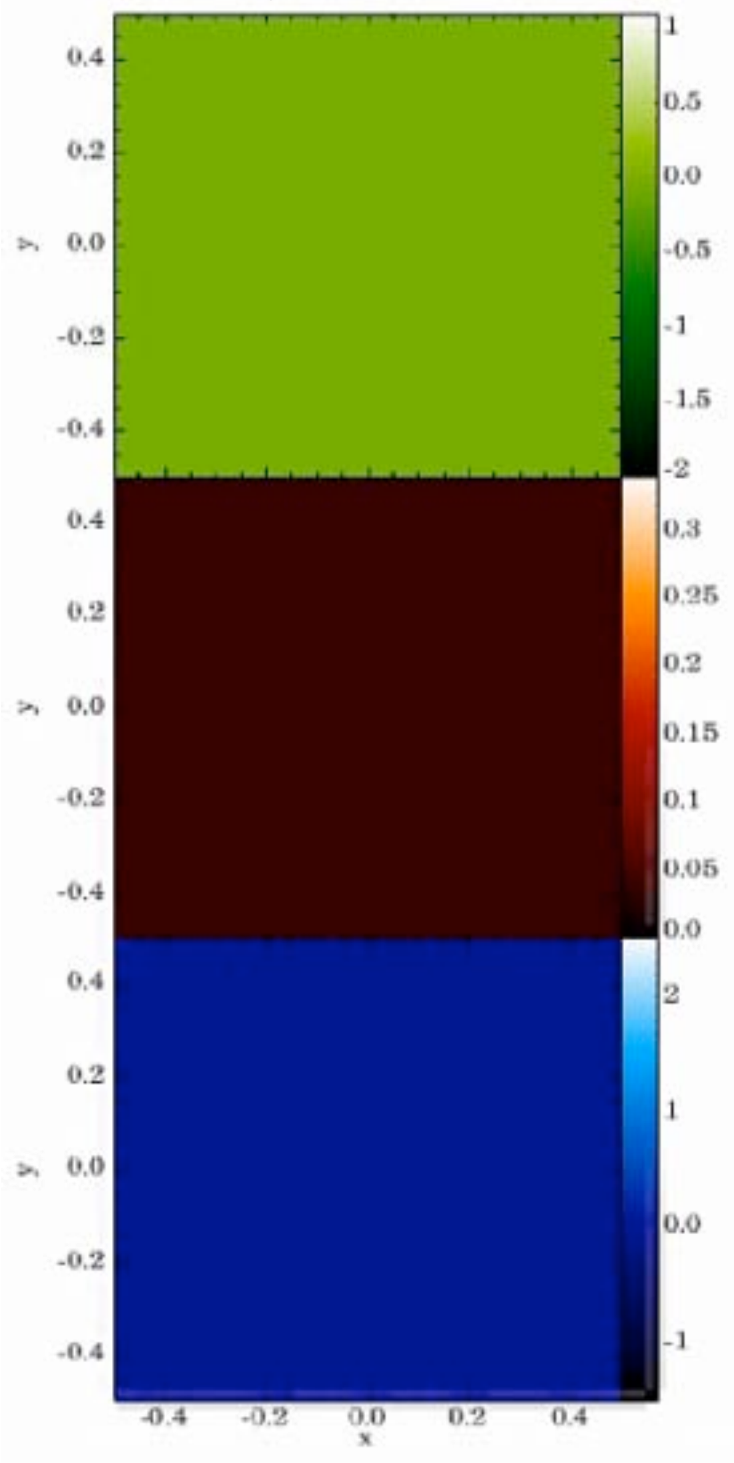
sound
speed

v_s

Toomre
parameter

Q

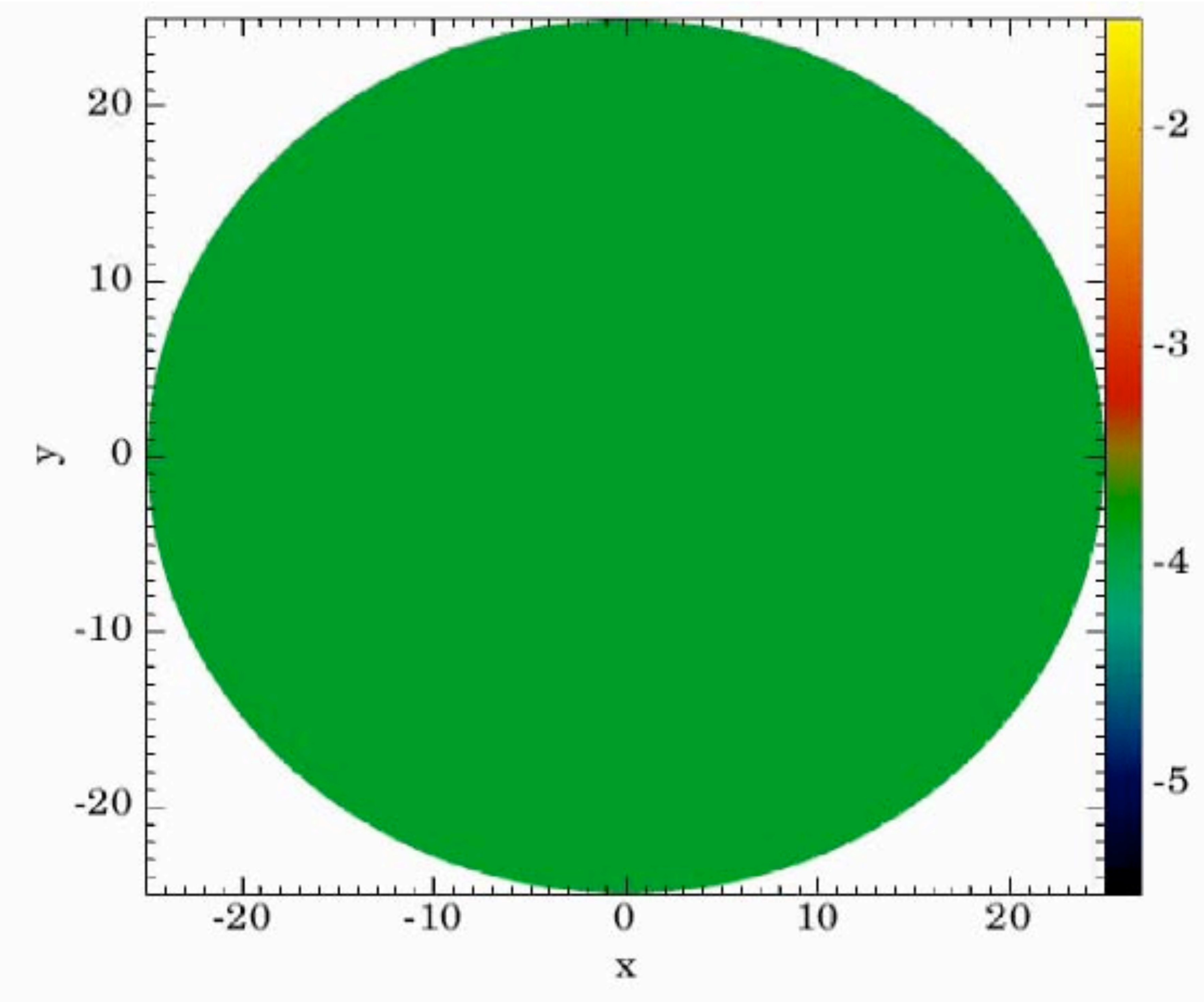
$\beta = 10$



Gravitational instability

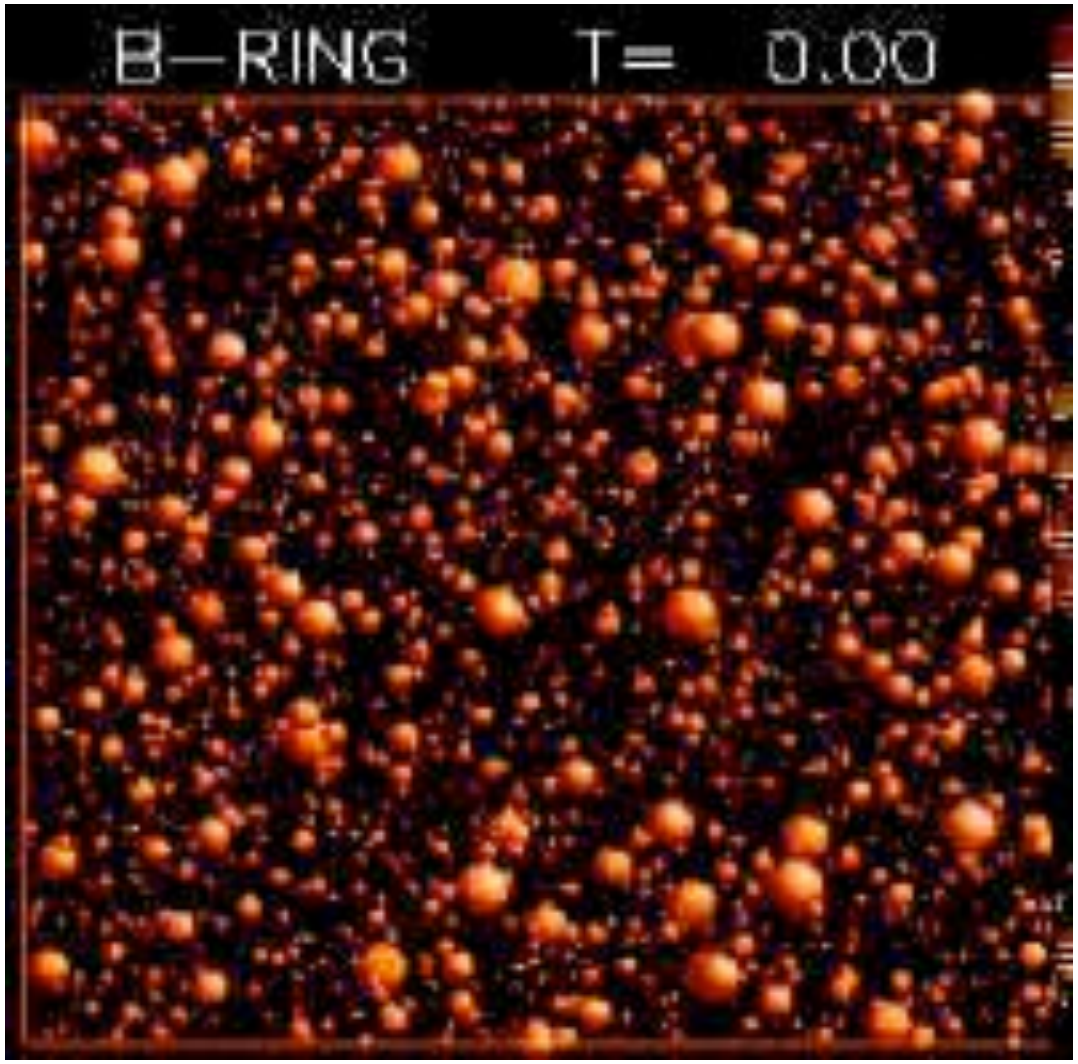
- 2D global simulation with cooling (S.-J. Paardekooper)

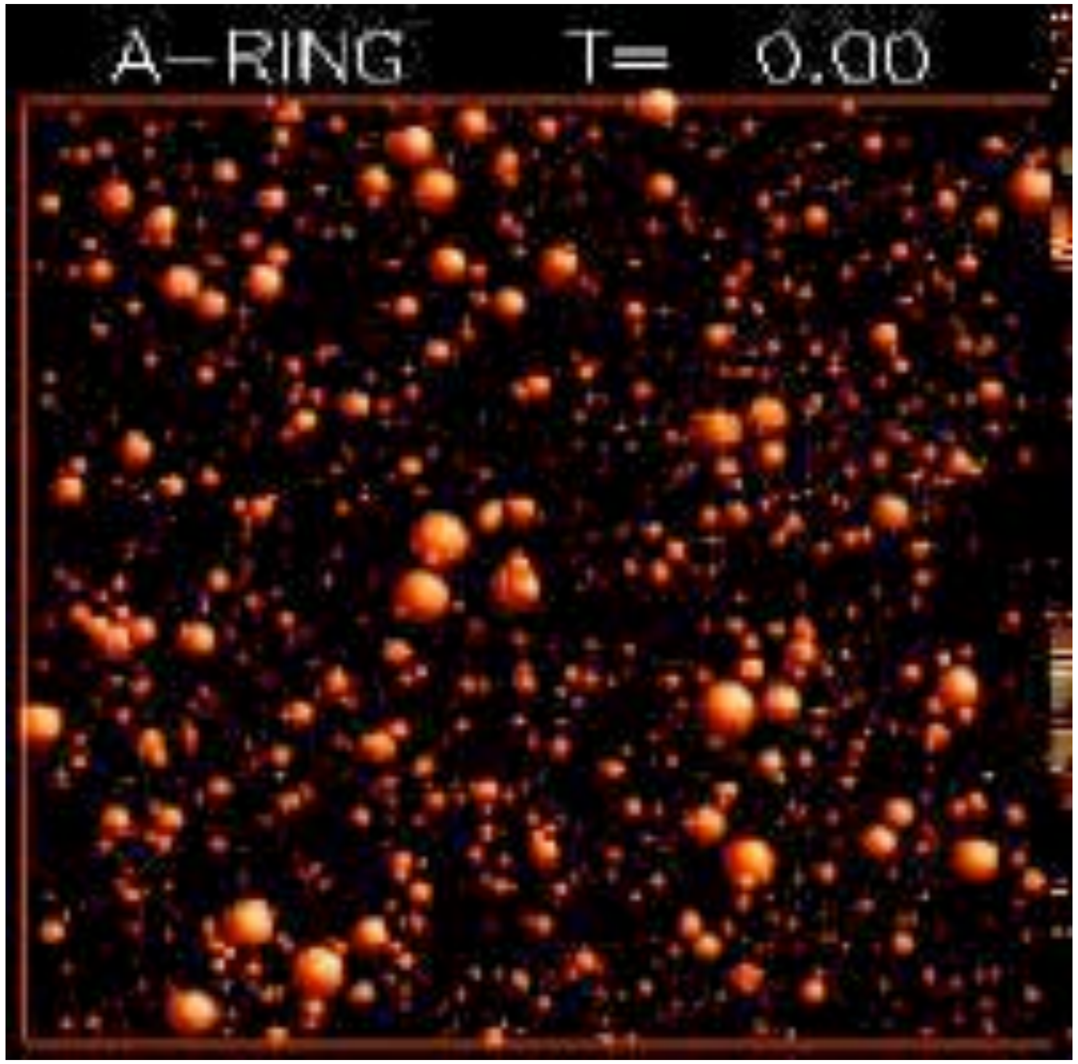
$\beta = 10$



Gravitational instability

- 3D simulations of Saturn’s rings (inelastically colliding particles) (Salo 1992)

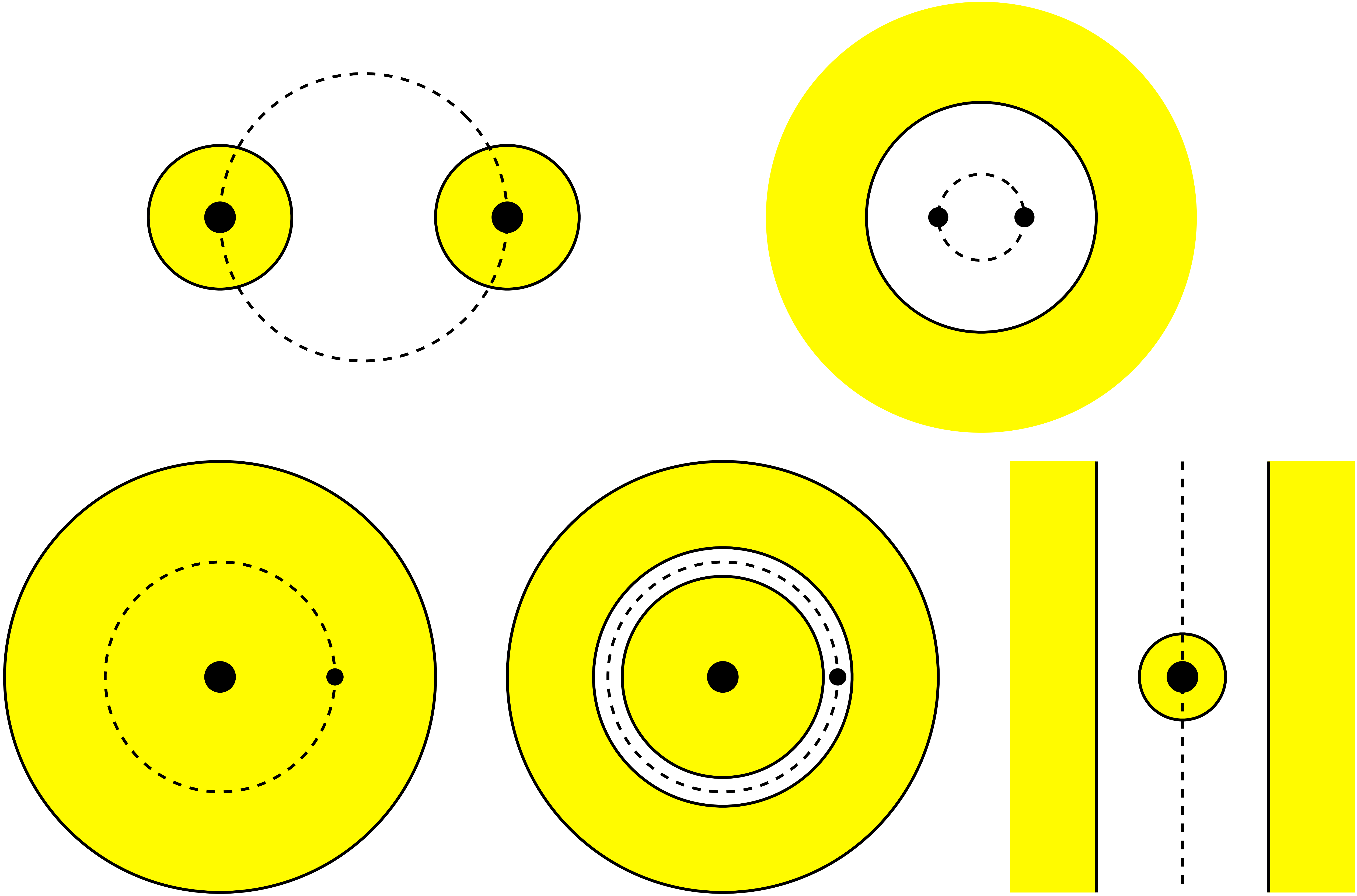




Satellite–disc interaction

- Common problem:
    - Orbiting companion, e.g. on circular orbit within disc
    - Gravitational (rather than hydrodynamic) interaction with disc
    - Perturbs orbital motion and excites waves
    - Calculate exchanges of energy and angular momentum
    - Determine orbital evolution of satellite (migration, etc.)

Satellite–disc interaction



Satellite–disc interaction

- Test particle dynamics in xy plane, in local approximation (fluid dynamics more difficult, but results are similar in some ways)

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial\Psi}{\partial x}$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Psi}{\partial y}$$

- Satellite on circular orbit at reference radius $(x_\mathrm{s} = y_\mathrm{s} = 0)$:

$$\Psi = -GM_\mathrm{s}(x^2 + y^2)^{-1/2}$$

Satellite–disc interaction

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial\Psi}{\partial x}$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Psi}{\partial y}$$

- General solution in absence of satellite potential:

$$\dddot{x} = -4\Omega^2\dot{x} + 2\Omega S\dot{x} = -\kappa^2\dot{x}$$

$$\Rightarrow \quad x = x_0 + A_\mathrm{r}\cos\kappa t + A_\mathrm{i}\sin\kappa t = x_0 + \mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$y = y_0 - S x_0 t - \frac{2\Omega}{\kappa}\,\mathrm{Re}\left[\mathrm{i}A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

- Guiding centre $(x_0, y_0 - S x_0 t)$
- Complex epicyclic amplitude $A = A_\mathrm{r} + \mathrm{i}A_\mathrm{i}$

Satellite–disc interaction

- To express “orbital elements” in terms of position and velocity:

$$x = x_0 + \mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$\dot{x} = \mathrm{Re}\left[-\mathrm{i}\kappa A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right] = \kappa\,\mathrm{Im}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$\ddot{x} = -\kappa^2\,\mathrm{Re}\left[A\,\mathrm{e}^{-\mathrm{i}\kappa t}\right]$$

$$\Rightarrow\quad A\,\mathrm{e}^{-\mathrm{i}\kappa t} = -\frac{\ddot{x}}{\kappa^2} + \frac{\mathrm{i}\dot{x}}{\kappa}$$

$$\Rightarrow\quad A = \left[-\frac{2\Omega}{\kappa^2}(\dot{y} + Sx) + \frac{\mathrm{i}\dot{x}}{\kappa}\right]\mathrm{e}^{\mathrm{i}\kappa t}$$

$$x_0 = x + \frac{\ddot{x}}{\kappa^2} = x + \frac{2\Omega}{\kappa^2}(\dot{y} + Sx) = \frac{2\Omega}{\kappa^2}(\dot{y} + 2\Omega x)$$

Satellite–disc interaction

- Canonical y momentum (per unit mass):

$$p_y = \dot{y} + 2\Omega x = \frac{\kappa^2}{2\Omega} x_0 = \mathrm{cst}$$

- Energy (per unit mass):

$$\varepsilon = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega S x^2$$

- Use $\kappa^2 |A|^2 = \dot{x}^2 + \dfrac{4\Omega^2}{\kappa^2}(\dot{y} + Sx)^2$:

$$\begin{aligned}
\varepsilon &= \frac{1}{2}\kappa^2 |A|^2 - \frac{2\Omega^2}{\kappa^2}(\dot{y} + Sx)^2 + \frac{1}{2}\dot{y}^2 - \Omega S x^2 \\
&= \frac{1}{2}\kappa^2 |A|^2 - \frac{\Omega S}{\kappa^2}(\dot{y} + 2\Omega x)^2 \\
&= \frac{1}{2}\kappa^2 |A|^2 - \frac{\Omega S}{\kappa^2} p_y^2 = \mathrm{cst}
\end{aligned}$$

Satellite–disc interaction

- In the presence of a satellite potential:

$$\dot{p}_y = -\frac{\partial\Psi}{\partial y}$$

$$\varepsilon + \Psi = \mathrm{cst}$$

$$\begin{aligned}\dot{A} &= \left[-\frac{2\Omega}{\kappa^2}(\ddot{y} + S\dot{x}) + \frac{\mathrm{i}\ddot{x}}{\kappa} - \frac{2\mathrm{i}\Omega}{\kappa}(\dot{y} + Sx) - \dot{x}\right] \mathrm{e}^{\mathrm{i}\kappa t} \\ &= \left[-\frac{2\Omega}{\kappa^2}(\ddot{y} + 2\Omega\dot{x}) + \frac{\mathrm{i}}{\kappa}(\ddot{x} - 2\Omega\dot{y} - 2\Omega Sx)\right] \mathrm{e}^{\mathrm{i}\kappa t} \\ &= \left(\frac{2\Omega}{\kappa^2}\frac{\partial\Psi}{\partial y} - \frac{\mathrm{i}}{\kappa}\frac{\partial\Psi}{\partial x}\right) \mathrm{e}^{\mathrm{i}\kappa t}\end{aligned}$$

Satellite–disc interaction

- Consider the unperturbed "circular" orbit ($A = 0$)

$$x = x_0 = \mathrm{cst}$$

$$y = -Sx_0t$$

- Calculate ΔA in linear approximation:

$$\dot{A} = \left(\frac{2\Omega}{\kappa^2}\frac{\partial\Psi}{\partial y} - \frac{\mathrm{i}}{\kappa}\frac{\partial\Psi}{\partial x}\right)\mathrm{e}^{\mathrm{i}\kappa t} \qquad \Psi = -GM_\mathrm{s}(x^2+y^2)^{-1/2}$$

$$= GM_\mathrm{s}(x^2+y^2)^{-3/2}\left(\frac{2\Omega y}{\kappa^2} - \frac{\mathrm{i}x}{\kappa}\right)\mathrm{e}^{\mathrm{i}\kappa t}$$

$$\approx -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}(1+S^2t^2)^{-3/2}\left(1 - \mathrm{i}\frac{2\Omega}{\kappa}St\right)\mathrm{e}^{\mathrm{i}\kappa t}$$

$$\Delta A = \int_{-\infty}^{\infty}\dot{A}\,\mathrm{d}t$$

$$= -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}\int_{-\infty}^{\infty}(1+S^2t^2)^{-3/2}\left(\cos\kappa t + \frac{2\Omega}{\kappa}St\sin\kappa t\right)\mathrm{d}t$$

Satellite–disc interaction

$$\Delta A = -\mathrm{i}\frac{GM_\mathrm{s}}{\kappa x_0^2}\int_{-\infty}^{\infty}(1+S^2t^2)^{-3/2}\left(\cos\kappa t+\frac{2\Omega}{\kappa}St\sin\kappa t\right)\mathrm{d}t$$

- Let $f(k)=\int_{-\infty}^{\infty}(1+x^2)^{-3/2}\cos kx\,\mathrm{d}x = 2k\underset{\uparrow}{K_1}(k) \quad (k>0)$

  modified Bessel function

- Then

$$\Delta A = -\mathrm{i}C\frac{GM_\mathrm{s}}{\kappa S x_0^2} \qquad C = f\left(\frac{\kappa}{S}\right)-\frac{2\Omega}{\kappa}f'\left(\frac{\kappa}{S}\right)$$

- For Keplerian orbits ($\kappa/S=2/3$), $\quad C\approx 3.359$
- So encounter with satellite excites an epicyclic oscillation at first order

Satellite–disc interaction

- Long before and after the encounter, $\Psi \to 0$
- Since $\varepsilon + \Psi$ is exactly conserved, $\Delta\varepsilon = 0$ in the encounter
- But $\varepsilon = \frac{1}{2}\kappa^2|A|^2 - \frac{\Omega S}{\kappa^2}p_y^2$, so $\Delta(p_y^2) = \frac{\kappa^4}{2\Omega S}\Delta(|A|^2)$
- Assume a “circular” orbit before the encounter:

$$A = 0, \quad p_y = \frac{\kappa^2}{2\Omega}x_0$$

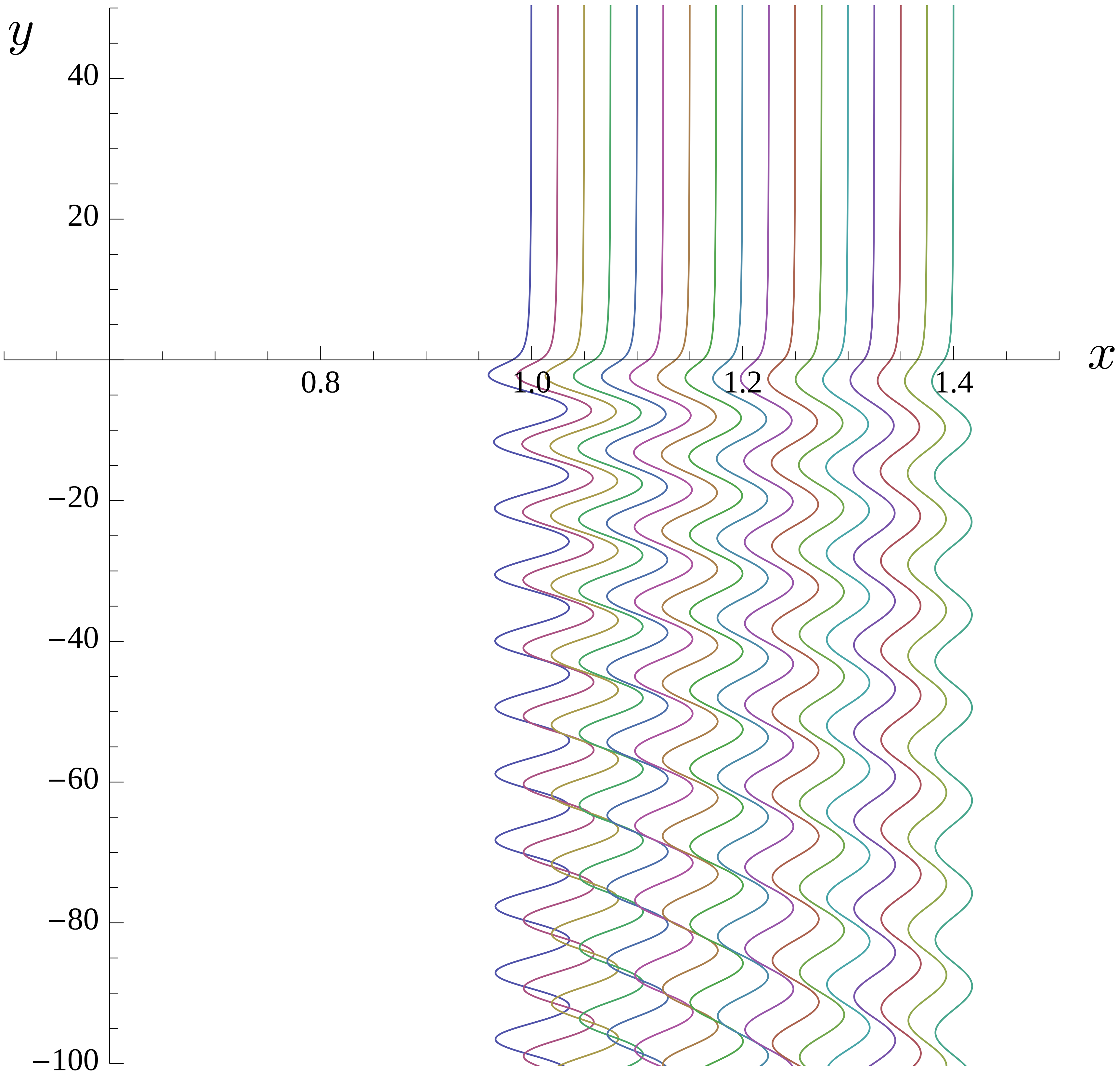
- Then, after the encounter:

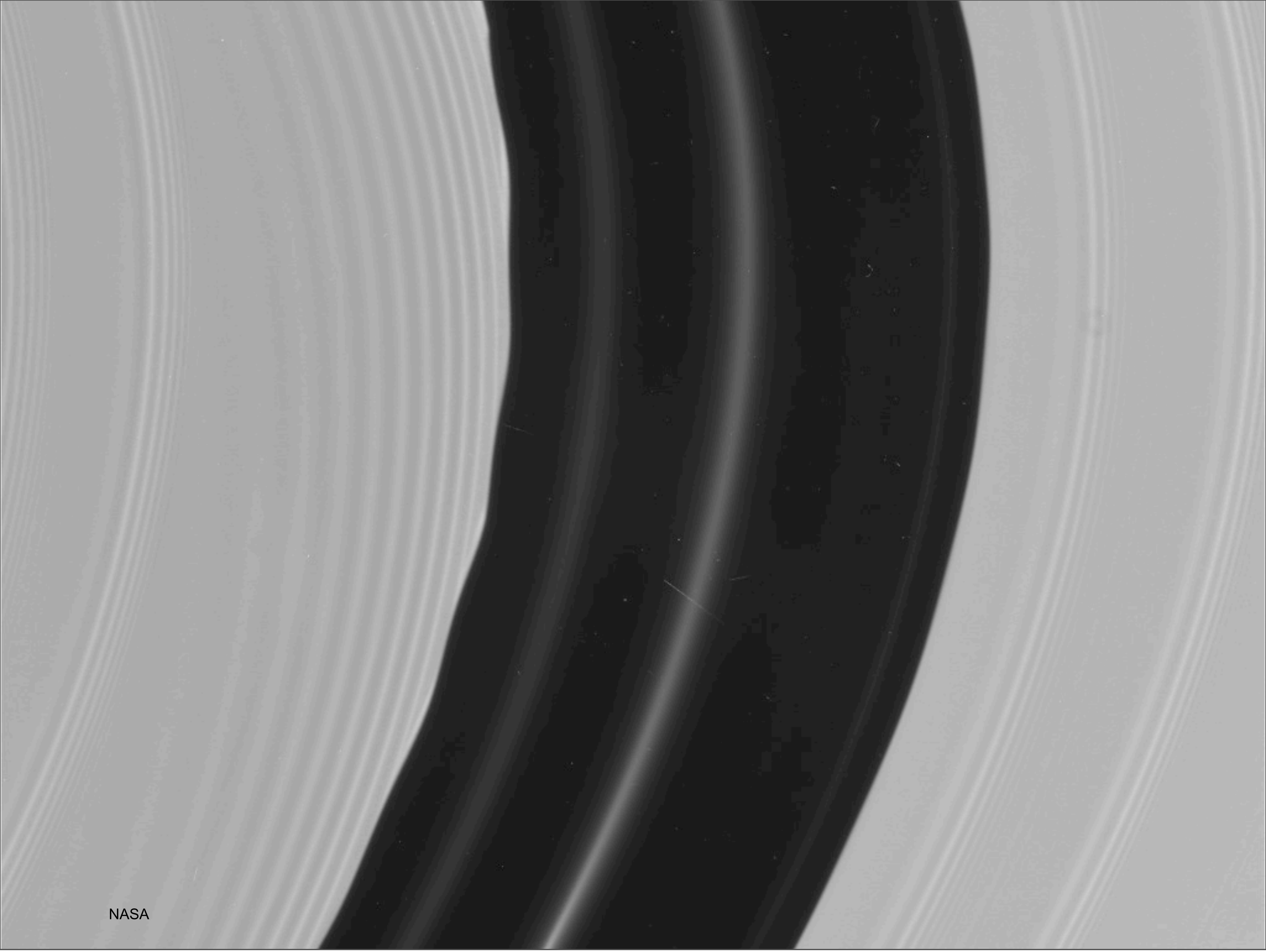
$$A \approx -\mathrm{i}C\frac{GM_\mathrm{s}}{\kappa S x_0^2}, \quad p_y^2 \approx \frac{\kappa^4}{4\Omega^2}x_0^2 + \frac{\kappa^4}{2\Omega S}\left(C\frac{GM_\mathrm{s}}{\kappa S x_0^2}\right)^2$$

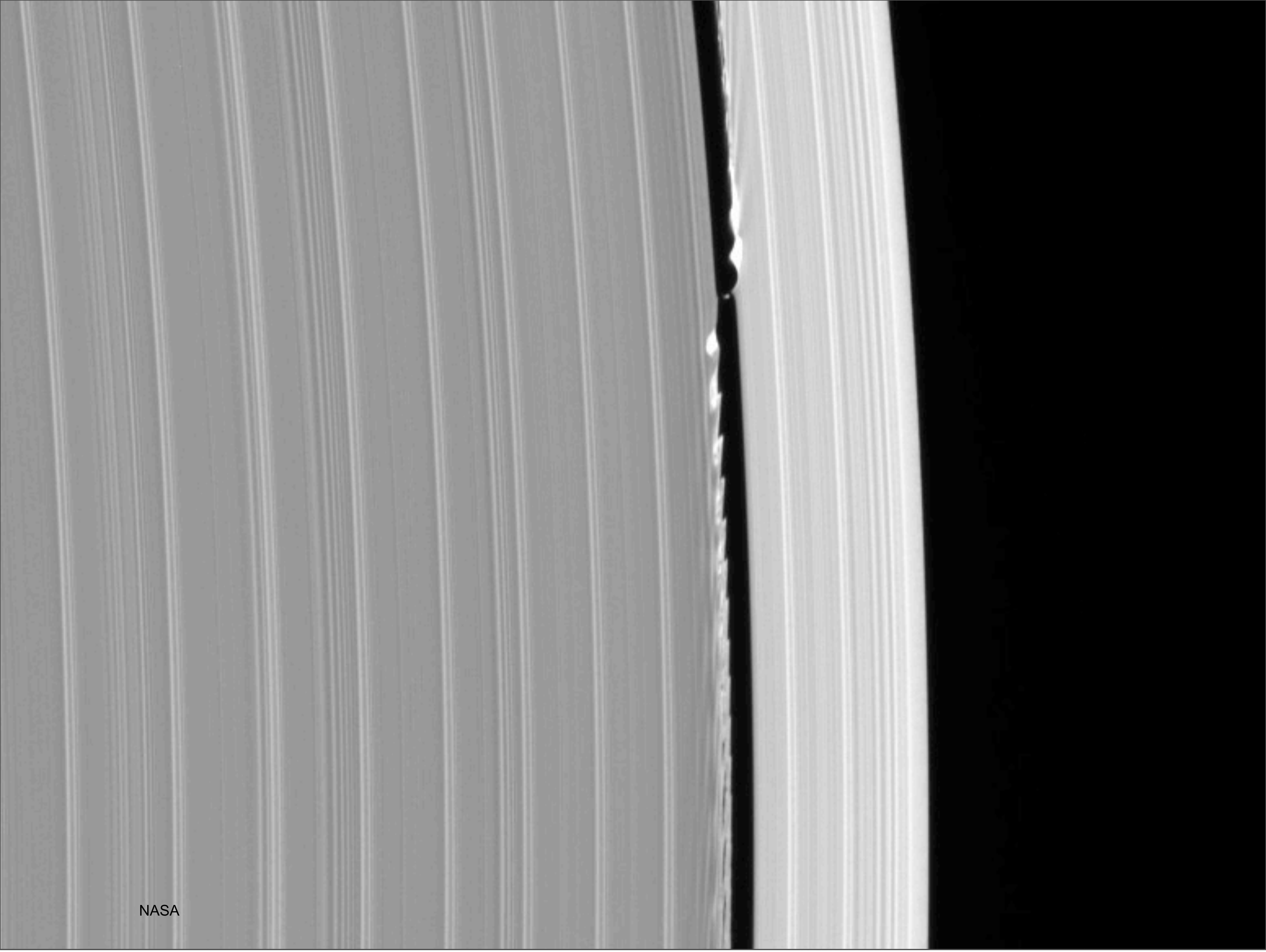
$$\Rightarrow \ p_y \approx \frac{\kappa^2}{2\Omega}x_0 + \underbrace{\frac{(CGM_\mathrm{s})^2}{2S^3x_0^5}}_{\Delta p_y \text{ correct to second order}}$$

- Irreversibility / dissipation implicit in assuming circular initial orbit

Satellite–disc interaction



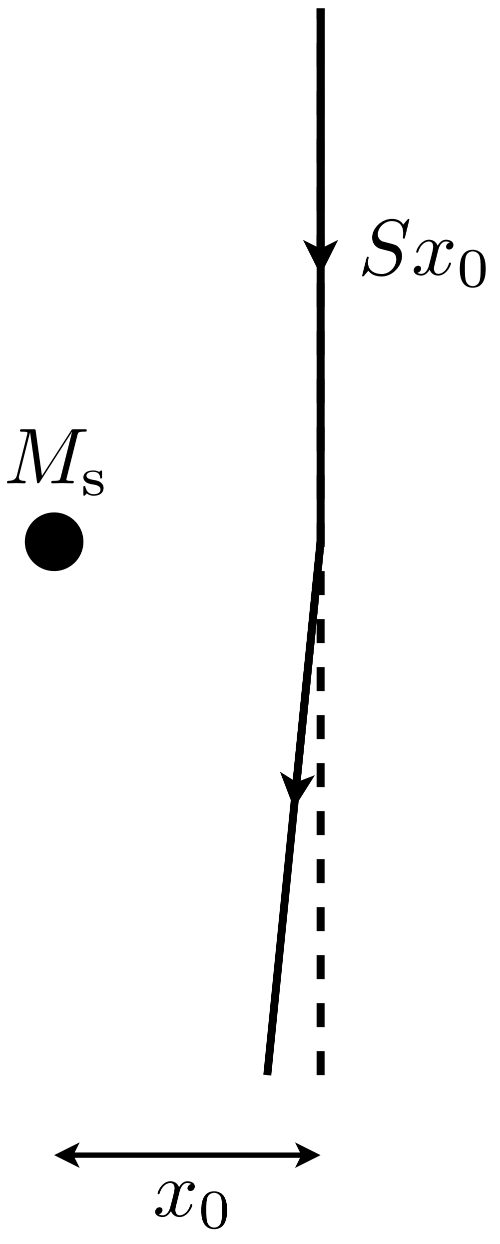

NASA



NASA

Satellite–disc interaction

- Simplified version: “impulse approximation”



$$\Delta v_\perp \approx \frac{GM_{\rm s}}{x_0^2}\frac{1}{S}$$

$$\Delta(v_\perp^2) + \Delta\left(v_\parallel^2\right) = 0 \quad \text{(energy)}$$

$$\left(\frac{GM_{\rm s}}{Sx_0^2}\right)^2 + 2Sx_0\Delta v_\parallel \approx 0$$

$$\Delta v_\parallel \approx -\frac{(GM_{\rm s})^2}{2S^3x_0^5} \quad \text{(lacks } C^2 \text{ factor)}$$

Satellite–disc interaction

- y force on disc per unit x at location x :

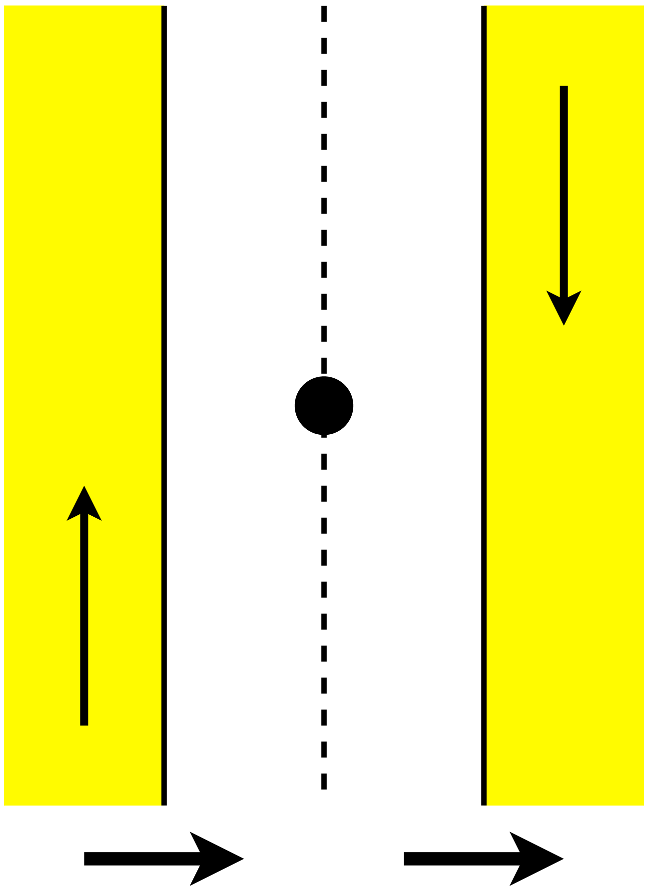
$$= \frac{(CGM_\mathrm{s})^2}{2S^3x^5} \underset{\substack{\big\uparrow \\ \text{surface density}}}{\Sigma} \underset{\substack{\uparrow \\ \text{encounter rate}}}{|Sx|}$$

$$\propto x^{-4}\,\mathrm{sgn}(x)$$

- Torque per unit radius is the same $\times r_0$
- Satellite experiences an equal an opposite torque
- Effect is of second order in M_s
- Similar result for density waves (response of a fluid disc)
- x^{-4} divergence is moderated within $|x| \lesssim H$ (or Hill radius)

Satellite–disc interaction

- Gravitational interaction is “repulsive”!



transfer of p_y (or angular momentum)

- One-sided torque leads to gap opening if M_s large enough and ν small enough
- Asymmetry leads to net torque on satellite and to migration (usually inwards)

Satellite–disc interaction

- Now include periodic nature of y coordinate $(L_y = 2\pi r_0)$:

$$\begin{aligned} \dot{A} &= \left(\frac{2\Omega}{\kappa^2} \frac{\partial \Psi}{\partial y} - \frac{\mathrm{i}}{\kappa} \frac{\partial \Psi}{\partial x} \right) \mathrm{e}^{\mathrm{i}\kappa t} \\ &= F(t)\, \mathrm{e}^{\mathrm{i}\kappa t} \\ &= \sum_{n=-\infty}^{\infty} f_n\, \mathrm{e}^{-\mathrm{i}n\omega t}\, \mathrm{e}^{\mathrm{i}\kappa t} \end{aligned}$$

$$T = \frac{2\pi r_0}{S|x_0|}$$

$$\omega = \frac{2\pi}{T} = \frac{S|x_0|}{r_0}$$

- Add damping of epicyclic motion:

$$\dot{A} = \sum_{n=-\infty}^{\infty} f_n\, \mathrm{e}^{-\mathrm{i}n\omega t}\, \mathrm{e}^{\mathrm{i}\kappa t} - \gamma A$$

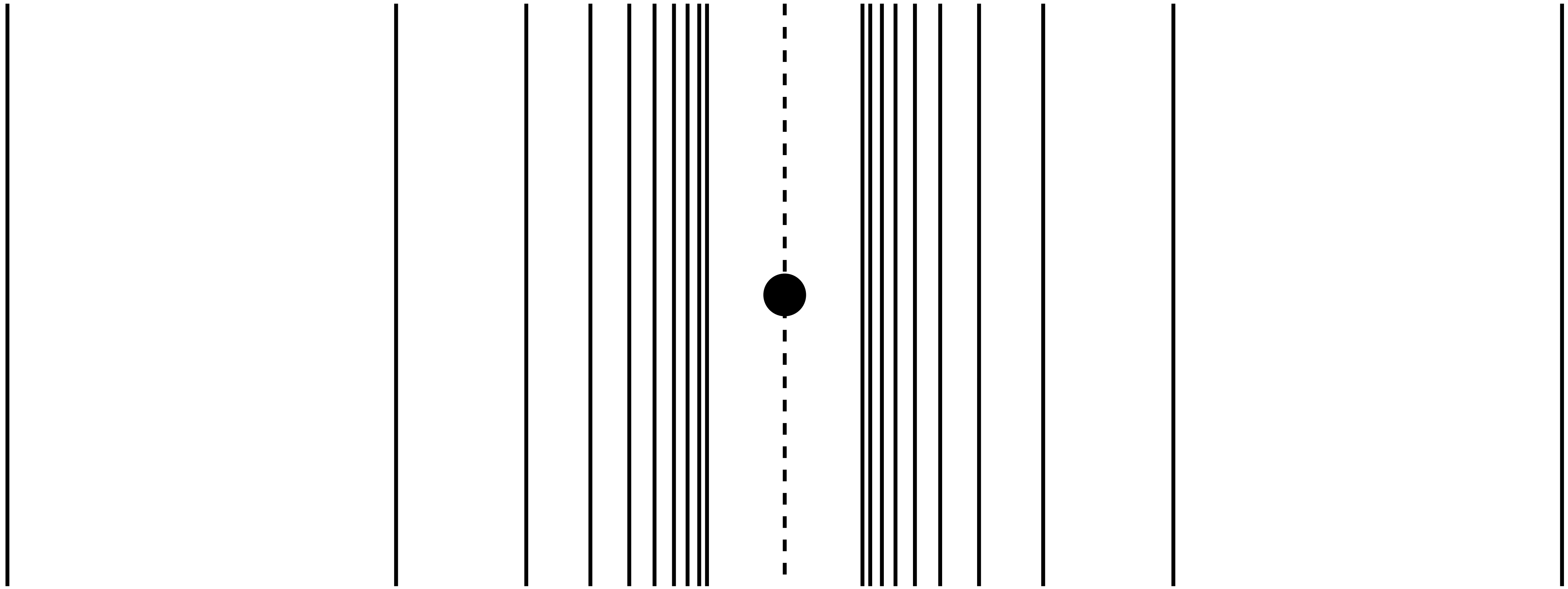
- Long-term response:

$$A = \sum_{n=-\infty}^{\infty} \frac{\mathrm{i} f_n\, \mathrm{e}^{-\mathrm{i}n\omega t}\, \mathrm{e}^{\mathrm{i}\kappa t}}{(n\omega - \kappa) + \mathrm{i}\gamma}$$

- “Lindblad resonances” where $\dfrac{x}{r_0} = \dfrac{1}{n}\dfrac{\kappa}{S}$, resolved by damping

Satellite–disc interaction

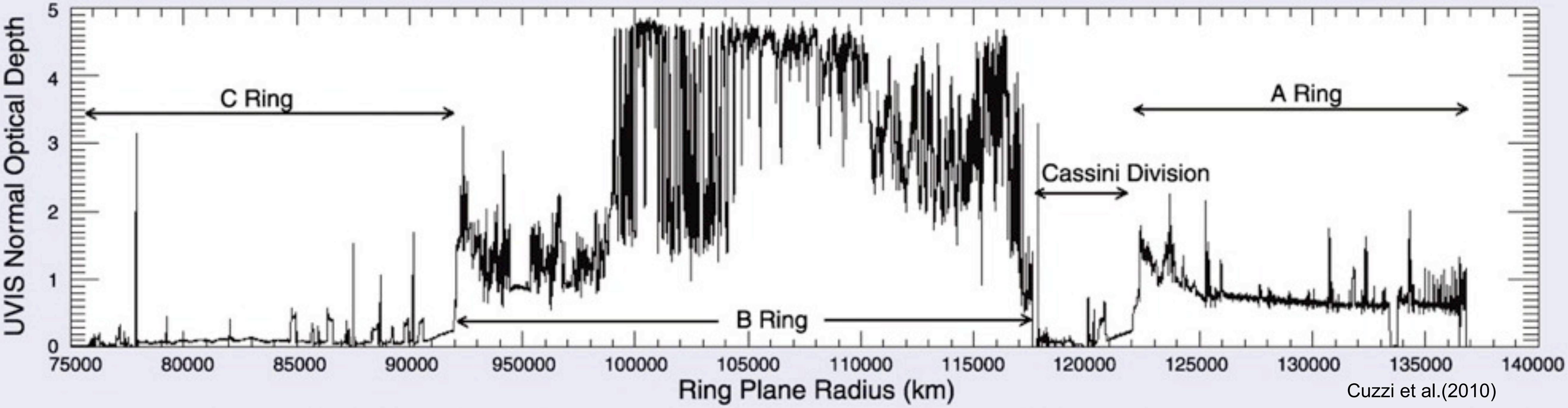
- Lindblad resonances: $\frac{x}{r_0} = \frac{1}{n}\frac{\kappa}{S}$



- In a Keplerian disc, LRs correspond to orbital commensurabilities

$$\frac{\Omega}{\Omega_0} = \frac{n}{n-1}$$

- In a fluid disc, density waves are launched there
(wave emission resolves singularity in response)



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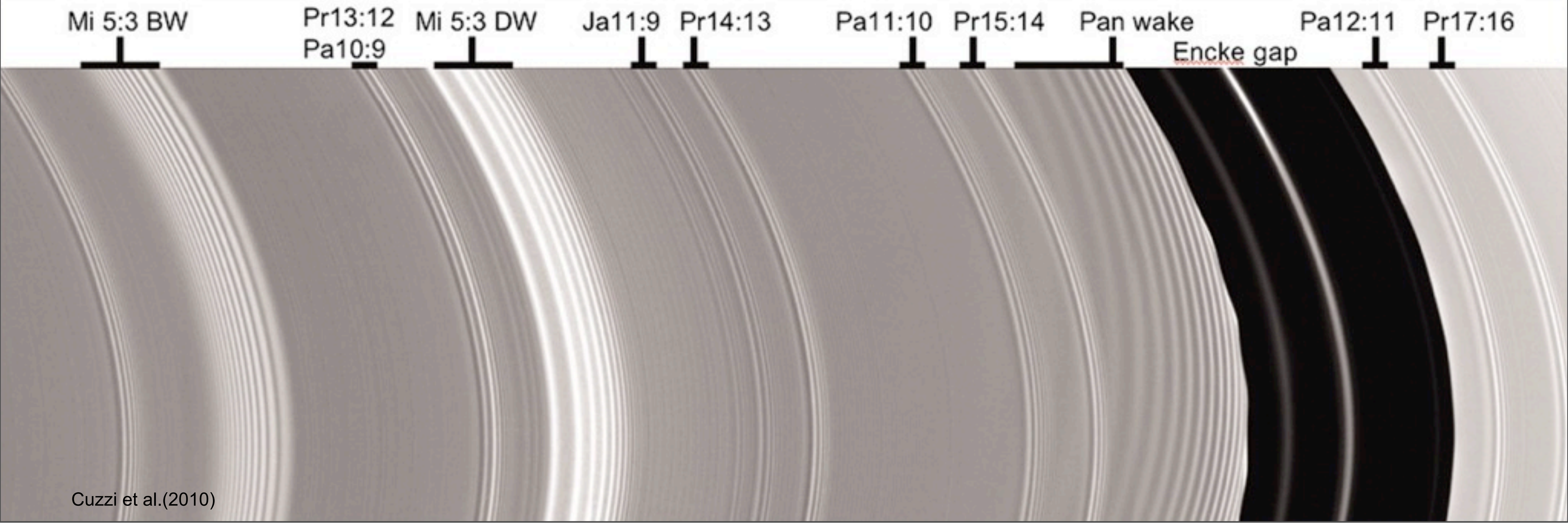

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