Fluid Dynamics in Earth and Planetary Sciences

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# Dynamics of geophysical vortices and jets

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# Lecture 1: Review of geophysical fluid dynamics

- Basic equations
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  - Examples

**1 —Basic equations**

I'll generally work with the adiabatic, inviscid equations and only introduced non-conservative terms as needed.

*1.1 — Mass*

The conservation of mass really defines $\mathbf{u}$ as the mass-weighted velocity of collection of molecules so that the flux is just $\mathbf{u}\rho$. Then

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{or} \quad \frac{1}{\rho}\frac{D}{Dt}\rho + \nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

with the Lagrangian (material) derivative defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

*1.2 — Momentum*

$$\frac{\partial}{\partial t}\mathbf{u} + (2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \frac{1}{2}|\mathbf{u}|^2 - \nabla \Phi \tag{1.2}$$

with the vorticity $\boldsymbol{\zeta}$ being

$$\boldsymbol{\zeta} = \nabla \times \mathbf{u} \tag{1.3}$$

The gravitational (including centrifugal) potential is $\Phi$ and the planet's rotation vector is $\boldsymbol{\Omega}$.

Often I'll use the Lagrangian derivative notation for velocities, but it really means

$$\frac{D}{Dt}\mathbf{u} = \frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} + \frac{1}{2}|\mathbf{u}^2|$$

in that context; in a non-Cartesian system, you need to worry about accelerations associated with changes in direction of the basis vectors.

*1.3 — Thermodynamics*

For adiabatic processes, the thermodynamics is most easily represented as the conservation of entropy $\eta$. If we think of the equation of state as specifying the density as a function of entropy and pressure (and salinity), we have the simple expression

$$\frac{D}{Dt}\rho = \frac{\partial \rho(\eta, p, S)}{\partial p}\frac{D}{Dt}p = \frac{1}{c_s^2}\frac{D}{Dt}p \tag{1.4}$$

with $c_s(p, \rho)$ being the speed of sound. For an ideal gas, $c_s^2 = \gamma RT = \gamma p/\rho$; in general, it's

$$c_s^2 = \left[\rho_p - \frac{T\rho_T^2}{c_p \rho^2}\right]^{-1}$$

for $\rho$ expressed in terms of the usual variables, $T$, $S$, $p$. For problems with non-conservative terms, it is more convenient to regard the enthalpy (internal energy$+p/rho$) as a primary variable. The thermodynamic quantities can be best summarized in terms of derivatives of the Gibb's function (internal energy$+p/\rho - T\eta$)

*1.4 — Vorticity*

The absolute vorticity $\boldsymbol{\zeta}_a = 2\boldsymbol{\Omega} + \boldsymbol{\zeta}$ evolves according to

$$\frac{\partial}{\partial t}\boldsymbol{\zeta}_a + \nabla \times (\boldsymbol{\zeta}_a \times \mathbf{u}) = \frac{1}{\rho^2}\nabla\rho \times \nabla p \tag{1.5}$$

(often written in the Cartesian form

$$\frac{\partial}{\partial t}\boldsymbol{\zeta}_a + (\mathbf{u}\cdot\nabla)\boldsymbol{\zeta}_a - (\boldsymbol{\zeta}_a \cdot \nabla)\mathbf{u} + \boldsymbol{\zeta}_a(\nabla\cdot\mathbf{u}) = \frac{1}{\rho^2}\nabla\rho \times \nabla p$$

to bring out the vorticity generation processes: stretching, tilting, and baroclinic). Our primary interest is in the local vertical component, so we shall instead dot 1.xx with $\hat{\mathbf{z}}$ to get an equation for the local vertical absolute vorticity $\zeta_a = \hat{\mathbf{z}} \cdot \boldsymbol{\zeta}_a = \zeta + f$.

$$\frac{\partial}{\partial t}\zeta_a + \nabla\cdot(\mathbf{u}\zeta_a) = \boldsymbol{\zeta}_a \cdot \nabla w + \frac{1}{\rho^2}\hat{\mathbf{z}}\cdot(\nabla\rho \times \nabla p) \tag{vort}$$

The r.h.s. has stretching terms from the vertical component of $\boldsymbol{\zeta}_a$, tilting terms from the horizontal components, and baroclinic generation.

To understand the meaning of the vorticity, consider the motion of a small line segment $\delta\mathbf{x}$ extending from $\mathbf{x}(t)$ to $\mathbf{x}(t) + \delta\mathbf{x}(t)$. At time $t + \delta t$, we find

$$\mathbf{x}(t+\delta t) = \mathbf{x}(t) + \mathbf{u}(\mathbf{x}(t), t)\delta t$$
$$\mathbf{x}(t+\delta t) + \delta\mathbf{x}(t+\delta t) = \mathbf{x}(t) + \delta\mathbf{x}(t) + \mathbf{u}(\mathbf{x}(t) + \delta\mathbf{x}(t), t)\delta t$$

or

$$\delta\mathbf{x}(t+\delta t) = \delta\mathbf{x}(t) + u(\mathbf{x}(t) + \delta\mathbf{x}(t), t)\delta t - \mathbf{u}(\mathbf{x}(t), t)\delta t$$

so that

$$\frac{d}{dt}\delta x_i = \frac{\partial u_i}{\partial x_j}\delta x_j$$

The nine-component rate-of-strain tensor can be split into symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

We can write the antisymmetric part in matrix form as

$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] = \frac{1}{2}\begin{pmatrix} 0 & -\zeta_z & \zeta_y \\ \zeta_z & 0 & -\zeta_x \\ -\zeta_y & \zeta_x & 0 \end{pmatrix}$$

But this is isomorphic to a rotation matrix: the product of this with a displacement vector takes the form

$$\begin{pmatrix} 0 & -\zeta_z & \zeta_y \\ \zeta_z & 0 & -\zeta_x \\ -\zeta_y & \zeta_x & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \boldsymbol{\zeta} \times \delta\mathbf{x}$$

Thus, the antisymmetric part of the rate-of-strain tensor contributes

$$\frac{d}{dt}\delta\mathbf{x} = \frac{1}{2}\boldsymbol{\zeta} \times \delta\mathbf{x} + \mathit{symm}$$

⇒ Vorticity causes line elements in the flow to rotate, with the vorticity vector being twice the local rate of rotation vector.

example    movement    divergence    strain    rotation    strain+rotation

The symmetric part of the rate-of-strain tensor stretches or shrinks line elements, corresponding to divergence — expansion of a volume element — and pure strain — expanding along one axis and contracting correspondingly along the other two (or vice-versa).

The rotation of the planet, $\boldsymbol{\Omega}$, ensures that the absolute vorticity $\boldsymbol{\zeta}_a = 2\boldsymbol{\Omega} + \boldsymbol{\zeta}$ is generally non-zero: **geophysical flows are inherently rotational**.

*1.5 — Streamfunction*

We shall at times use the streamfunction and velocity potential; these decompose the flow into its irrotational and rotational parts

$$\mathbf{u} = -\nabla\varphi - \nabla \times \vec{\psi}$$

which implies

$$\nabla^2\varphi = -\nabla \cdot \mathbf{u}$$

and

$$\nabla^2\vec{\psi} = \nabla \times \mathbf{u}$$

(if we choose $\nabla \cdot \vec{\psi} = 0$). Thus we can find the velocity potential and the streamfunction if we know the divergence and the vorticity.

## 2 —Approximations

*2.1 — Anelastic*

Our first simplification filters out sound waves. We let

$$\rho = \frac{\overline{\rho}(p)}{1+\tau}$$

where $\overline{\rho}(p)$ corresponds to an isentropic, hydrostatic state

$$\nabla \overline{p} = -\overline{\rho} \nabla \Phi$$

Think of $\tau$ as acting like the Boussinesq $\alpha T$. The pressure and gravity terms on the momentum equation become

$$\frac{1}{\rho}\nabla p + \nabla \Phi = (1+\tau)\nabla \int^p \frac{dp'}{\overline{\rho}(p')} + \nabla\Phi = \nabla\phi - \tau\nabla\Phi + \tau\nabla\phi \tag{2.1}$$

where we have removed the isentropic, hydrostatic balance

$$\int^p \frac{dp'}{\overline{\rho}(p')} = -\Phi + \phi \tag{2.2}$$

We approximate by dropping the last term in (2.xx) giving momentum equations

$$\frac{\partial}{\partial t}\mathbf{u} + (2\mathbf{\Omega} + \boldsymbol{\zeta}) \times \mathbf{u} = -\nabla\phi - \nabla\frac{1}{2}|\mathbf{u}|^2 + \tau\nabla\Phi \tag{2.3}$$

in which the buoyancy $\tau\nabla\Phi$ appears naturally.

We drop the $\tau$ term in the mass equation

$$\frac{D}{Dt}\ln\rho + \nabla\cdot\mathbf{u} = \frac{D}{Dt}\ln\overline{\rho} - \frac{D}{Dt}\ln(1+\tau) + \nabla\cdot\mathbf{u} = 0 \quad \rightarrow \quad \nabla\cdot\overline{\rho}\mathbf{u} = 0 \tag{2.4}$$

From the definition of $\phi$, the thermodynamic equation

$$\frac{D}{Dt}p = c_s^2 \frac{D}{Dt}\rho$$

becomes

$$\frac{D}{Dt}\phi - \mathbf{u}\dot{\nabla}\Phi = c_s^2\frac{1}{\overline{\rho}}\rho\frac{D}{Dt}\ln\rho = c_s^2(\overline{\rho}/(1+\tau), p)\frac{1}{1+\tau}\left(\frac{1}{\overline{\rho}}\mathbf{u}\cdot\nabla\overline{\rho} - \frac{1}{1+\tau}\frac{D}{Dt}\tau\right)$$

We use

$$\mathbf{u}\cdot\nabla\overline{\rho} = \frac{1}{c_s^2(\overline{\rho},\overline{p})}\mathbf{u}\cdot\nabla\overline{p} = -\frac{\overline{\rho}}{c_s^2(\overline{\rho},\overline{p})}\mathbf{u}\cdot\nabla\Phi$$

and drop the $\tau$'s compared to 1 to arrive at

$$\frac{D}{Dt}\tau = -\frac{1}{c_s^2}\frac{D}{Dt}\phi$$

Taking small Mach number again yields

$$\frac{D}{Dt}\tau = 0 \tag{2.5}$$

We collect (2.3,4,5) as our equation set

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u} + (2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \times \mathbf{u} &= -\nabla\phi - \nabla\frac{1}{2}|\mathbf{u}|^2 + \tau\nabla\Phi \\
\nabla \cdot \overline{\rho}\mathbf{u} &= 0 \qquad\qquad (Anelastic) \\
\frac{D}{Dt}\tau &= 0
\end{aligned}$$

Note that, with no-flux or periodic boundary conditions, these equations conserve energy

$$\frac{\partial}{\partial t}\int d\mathbf{x}\left(\frac{1}{2}\overline{\rho}|\mathbf{u}|^2 + \overline{\rho}\tau\Phi\right) = 0$$

Note on $\tau$: if we think of $\rho$ as a function of potential temperature $\theta$, salinity $S$, and pressure $p$, we have

$$\tau = \frac{\rho(\theta_0, S_0, p) - \rho(\theta, S, p)}{\rho(\theta, S, p)}$$

with $\theta' = \theta - \theta_0$ and $S' = S - S_0$ being conserved. For an ideal gas, $\eta = c_p \ln\theta = c_v \ln p - c_p \ln\rho$ and $\rho = Kp^{c_v/c_p}/\theta$. The equation above simplifies to

$$\tau = \frac{1/\theta - 1\theta_0}{1/\theta} = \frac{\theta - \theta_0}{\theta_0} = \frac{\theta'}{\theta_0}$$

Thus $\tau$ is conserved and is a surrogate for entropy. In the more general case, we can expand to find

$$\tau \simeq \alpha\theta' - \beta S' \quad \text{with} \quad \alpha = -\frac{\partial}{\partial\theta}\ln\rho \quad , \quad \beta = \frac{\partial}{\partial S}\ln\rho$$

evaluated at $\theta_0$, $S_0$. In the Boussinesq approximation, these are taken to be constant so $\tau$ is again conserved. But in general, they will be functions of pressure so that you get extra terms in the thermodynamic equation, related to the $p$ derivatives of the expansion coefficients. The baroclinic term in the PV derivation

$$\nabla\eta \cdot \nabla \times \tau\Phi = \nabla\eta \cdot (\nabla\tau \times \nabla\Phi)$$

vanishes when we can use $\tau$ as the conserved property in place of $\eta$ but not otherwise.

*2.2 — Hydrostatic and traditional approximation*

For most of the lectures, I'll use the hydrostatic approximation, appropriate for systems with the horizontal scale large compared to the vertical scale. However, it may not be apply to deep atmospheres on the gas giants.

Central to the approximation is that the vertical velocity $w$ is order $H/L$ compared to the horizontal velocities $\mathbf{u}_h$ based on the mass equation. The horizontal vorticities are dominated by the vertical shear of the horizontal velocities. E.g., in Cartesian form $w_x \sim UH/L^2$ and is order $H^2/L^2$ compared to the other term $u_z \sim U/H$. In spherical coordinates, we also assume $L$ is less than or order of the planetary radius $a$. The horizontal equations become

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\mathbf{f} + \boldsymbol{\zeta}_h) \times \mathbf{u} = -\nabla\phi - \nabla\frac{1}{2}|\mathbf{u}_h|^2$$

with $\boldsymbol{\zeta}_h = \nabla \times \mathbf{u}_h$. The $w$ term still appears since $w\frac{\partial}{\partial z}u$ is similar in order to $u\frac{\partial}{\partial x}u$. Note that we have dropped the $w$ terms from $2\mathbf{\Omega}\times\mathbf{u}$; to be energetically consistent, we also drop the Coriolis term in the vertical momentum equation. Thus we replace $2\mathbf{\Omega}$ by $2(\mathbf{\Omega}\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} = \mathbf{f}$ with $\mathbf{f} = f\hat{\mathbf{z}}$ and the Coriolis parameter, $f$, being $f = 2\Omega\sin\theta$.

The vertical momentum equation has terms of the order listed

$$\begin{array}{cccc}
\dfrac{D}{Dt}w & -\,2\Omega\cos\theta u = & -\dfrac{\partial}{\partial z}\phi & +\,g\tau \\
\dfrac{U}{L}\dfrac{UH}{L} & fU & \dfrac{U^2}{H},\dfrac{fUL}{H} & ? \\
\dfrac{H^2}{L^2} & \dfrac{H/L}{Ro} & 1 & \tau \sim U^2/gH \\
Ro\dfrac{H^2}{L^2} & \dfrac{H}{L} & 1 & \tau \sim fUL/gH
\end{array}$$

where we have listed two possible scalings for $\phi$ and the implications for the ratio. The Rossby number $Ro = U/fL$ measures the importance of the inertia to the Coriolis force, or, alternatively, the size of an inertial circle† compared to the scale of the motion. In each case the acceleration is down by a factor of at least $H^2/L^2$. As mentioned, we need to drop the Coriolis term; that works out if $H/L << Ro$. That's OK, since the $U^2$ pressure scaling is appropriate for $ro$ order 1 while the $fUL$ applies for small $Ro$.

The three momentum equations now take the form

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\mathbf{f} + \boldsymbol{\zeta}_h) \times \mathbf{u} = -\nabla\phi - \nabla\frac{1}{2}|\mathbf{u}_h|^2 + \tau\nabla\Phi \qquad (hydrostat)$$

The other part of the traditional approximation involves writing $r = a + z$ and dropping all $H/a$ terms; this means terms like longitudinal derivatives

$$\frac{1}{r\cos\theta}\frac{\partial}{\partial\lambda} \to \frac{1}{a\cos\theta}\frac{\partial}{\partial\lambda}$$

† $\frac{\partial^2}{\partial t^2}\mathbf{u} + f^2\mathbf{u} = 0 \quad\Rightarrow\quad u = U\cos(ft) \quad\Rightarrow\quad X = \frac{U}{f}\sin ft$

or the vertical derivatives in the mass equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho w \rightarrow \frac{\partial}{\partial z}\rho w$$

change slightly.

*2.3 — Geostrophic*

### Pressure and flow

The geostrophic approximation starts with the assumption that the time scale 1s the advective time. The horizontal momentum equations scale like

$$\begin{array}{ccccc} \frac{\partial}{\partial t}\mathbf{u}_h & + (\mathbf{f} + \boldsymbol{\zeta}_h) \times & \mathbf{u} = & -\nabla\phi & - \nabla\frac{1}{2}|\mathbf{u}_h|^2 \\ U^2/L & fU & U^2/L & ? & U^2/L \\ Ro & 1 & Ro & 1 & Ro \end{array}$$

If $f >> U/L$, we indeed need to scale the pressure by $fUL$, and the ratios become of the order in the third line above. For small Rossby number, we find

$$\mathbf{u}_h = \frac{1}{f}\hat{\mathbf{z}} \times \nabla\phi$$

This is generally coupled with the hydrostatic equation, giving the thermal wind balance

$$f\frac{\partial}{\partial z}\mathbf{u}_h = \hat{\mathbf{z}} \times \nabla\frac{\partial\phi}{\partial z} = g\hat{\mathbf{z}} \times \nabla\tau$$

Note: we can solve for the balanced condition in general for a spherical planet with purely zonal flow $u = u(r, \theta)$; we find the equivalent of the cyclo-geostrophic equation

$$2\Omega\frac{d}{dz}\rho u + \frac{1}{r\cos\theta}\frac{d}{dz}\rho u^2 = \frac{g(r)}{r}\frac{\partial\rho}{\partial\theta}$$

with

$$\frac{d}{dz} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

The shears related to the density gradients are along the axis of rotation, not the vertical direction and, for the Coriolis term, involve the momentum rather than the velocity.

While these approximations simplify the dynamics by filtering out various kinds of motion, they imply that some fields are only known diagnostically: in particular, the geostrophic and hydrostatic equation relates the velocities and buoyancy to the pressure, but does not predict the evolution.

## 3 —Ertel's theorem

For the anelastic, hydrostatic system, the potential vorticity

$$q = \frac{1}{\overline{\rho}} \boldsymbol{\zeta}_a \cdot \nabla \tau$$

(with $\boldsymbol{\zeta}_a = \boldsymbol{\zeta}_h + \mathbf{f}$) is conserved, $\frac{D}{Dt} q = 0$.[†] Physically, this follows from conservation of circulation around a cylinder embedded between two adjacent $\tau$ surfaces with fixed difference $\delta\tau$.

$$\mathcal{C} = \oint \boldsymbol{\zeta}_a \cdot d\ell = A\boldsymbol{\zeta}_a \cdot \hat{\mathbf{n}} = A\boldsymbol{\zeta}_a \cdot \nabla\tau \frac{1}{|\nabla\tau|}$$

with the mass being

$$\mathcal{M} = \overline{\rho} A \frac{\delta\tau}{|\nabla\tau|}$$

Conservation of $\mathcal{C}/\mathcal{M}$ leads to Ertel's theorem.

Proof: uses the following vector identities:

- $\nabla\tau \cdot \nabla \times \mathbf{a} = \nabla \cdot (\mathbf{a} \times \nabla\tau)$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$
- $\nabla \cdot (a\mathbf{u}) = \mathbf{u} \cdot \nabla a + a\nabla \cdot \mathbf{u}$
- $\nabla \cdot \boldsymbol{\zeta}_a = \nabla \cdot (\nabla \times \mathbf{u}_h + \mathbf{f}) = 0$

[no need to introduce things like $(\boldsymbol{\zeta}_a \cdot \nabla)\mathbf{u}$ ].

$$\begin{aligned}
\frac{\partial}{\partial t} q &= \frac{1}{\overline{\rho}} \nabla\tau \cdot \frac{\partial}{\partial t}\boldsymbol{\zeta}_a + \frac{1}{\overline{\rho}}\boldsymbol{\zeta}_a \cdot \nabla \frac{\partial \tau}{\partial t} \\
&= -\frac{1}{\overline{\rho}} \nabla\tau \cdot (\nabla \times [\boldsymbol{\zeta}_a \times \mathbf{u}]) + \frac{1}{\overline{\rho}} \nabla\tau \cdot \nabla \times g\tau\hat{\mathbf{z}} - \frac{1}{\overline{\rho}} \boldsymbol{\zeta}_a \cdot \nabla(\mathbf{u} \cdot \nabla\tau) \\
&= -\frac{1}{\overline{\rho}} \nabla \cdot (\overline{\rho}\mathbf{u}q) + \frac{1}{\overline{\rho}} \nabla \cdot [\boldsymbol{\zeta}_a(\mathbf{u} \cdot \nabla\tau)] - \frac{1}{\overline{\rho}} \nabla \cdot [\boldsymbol{\zeta}_a(\mathbf{u} \cdot \nabla\tau)] + \frac{1}{\overline{\rho}} \nabla\tau \cdot \nabla \times g\tau\hat{\mathbf{z}} \\
&= -\frac{1}{\overline{\rho}} \nabla \cdot (\overline{\rho}\mathbf{u}q) + \frac{1}{\overline{\rho}} \nabla\tau \cdot \nabla \times g\tau\hat{\mathbf{z}} \\
&= -\mathbf{u} \cdot \nabla q - \frac{1}{\overline{\rho}} \nabla\tau \cdot (\hat{\mathbf{z}} \times \nabla g\tau)
\end{aligned}$$

The last term is zero since $g$ depends only on $z$. In the full Navier-Stokes equations, we require

$$\nabla\eta \cdot (\nabla p \times \nabla\rho) = 0$$

with $\eta$ the entropy; this is true for $\rho = \rho(\eta, p)$ but will not hold exactly when salinity is considered.

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[†] We'll use "PV" for potential vorticity.

## 4 —Barotropic vorticity equation

We can learn a lot about vortex and jet dynamics by studying the barotropic vorticity equation. Consider an isentropic fluid so that $\tau = 0$. We also drop the $\overline{\rho}$ term, consistent with very large $c_s^2$.

$$\nabla \cdot \mathbf{u} = 0$$

(If we think of this as dropping the $\frac{D}{Dt} \ln \rho$ in the mass equation, we cannot turn around and say $\nabla \cdot \mathbf{u} = 0$ so $\frac{D}{Dt}\rho = 0$.) We can also take $w = 0$ and $\frac{\partial}{\partial z}\mathbf{u}_h = 0$ if we use the hydrostatic/traditional approximation: the horizontal vorticity is then zero and stays zero.

Since the velocity potential will be zero,

$$\mathbf{u}_h = -\nabla \times \psi \hat{\mathbf{z}} = \hat{\mathbf{z}} \times \nabla \psi \quad , \quad \zeta = \nabla^2 \psi$$

and the vertical vorticity equation reduces to the conservation of "potential vorticity" (PV)

$$\frac{\partial}{\partial t} q + [\psi, q] = 0$$

$$q = \zeta + f \qquad (BTVE)$$

or

$$\nabla^2 \psi = q - f$$

with $[A, B] = \hat{\mathbf{z}} \cdot (\nabla A \times \nabla B)$ often written as $J(A, B)$. This equation embodies a fundamental approach to GFD: it has a conserved scalar $q$ which can be inverted to find the flow $\psi$. We can write the inversion formula in terms of the Green's function

$$\psi(\mathbf{x}) = \int d\mathbf{x}' G(\mathbf{x} - \mathbf{x}') \Big( q(\mathbf{x}') - f(\mathbf{x}') \Big) \quad , \quad \nabla^2 G = \delta(\mathbf{x} - \mathbf{x}') \qquad (BT - inv)$$

## 5 —Layer models

For stratified problems, we will use the two-layer model (Phillips, 1951): in order to study baroclinic processes, we need at least two degrees of freedom in the vertical. The traditional derivation deals with two layers of constant density, light fluid on top of a constant density, heavier layer. For atmospheres, however, we shall work instead with the assumption that we have a isentropic buoyant layer $\tau = g'/g$ overlaying the deeper isentropic layer.

$$\tau = \frac{g'}{g}\mathcal{H}(r+h)$$

This leads to

$$\phi_1 = \phi_1' + g'(r-a) \quad , \quad \phi_2 = \phi_2'$$

giving

$$\phi_1' = \phi_2' + g'h \quad \text{at } r = a-h$$

If the upper layer is hydrostatic, then the horizontal gradients are just

$$\nabla\phi_1' = \nabla\Big(\phi_2'(\mathbf{x}_h, a-h) + g'h(\mathbf{x}_h)\Big)$$

and, if deep layer is still thin enough that it also is hydrostatic, we just have

$$\frac{\partial}{\partial t}\mathbf{u}_1 + (f+\zeta)\hat{\mathbf{z}}\times\mathbf{u}_1 = -\nabla\left(\phi_2' + g'h + \frac{1}{2}\mathbf{u}_1^2\right)$$
$$\frac{\partial}{\partial t}\mathbf{u}_2 + (f+\zeta)\hat{\mathbf{z}}\times\mathbf{u}_2 = -\nabla\left(\phi_2' + \frac{1}{2}\mathbf{u}_2^2\right)$$

for the horizontal momentum equations. Again, it is consistent for the horizontal vorticity to vanish, so that $\mathbf{u}_1$ and $\mathbf{u}_2$ are horizontal.

Using $\frac{\partial}{\partial z}\mathbf{u}_h = 0$ and the fact that $z = -h$ is a material surface enables the integration of the mass equation

$$\left(\int_{-h}^{0}\overline{\rho}\right)\nabla\cdot\mathbf{u}_1 + \overline{\rho}(-h)\frac{d}{dt}h = 0$$

to be written as

$$\frac{D}{Dt}\ln\int_{-h}^{0}\overline{\rho} + \nabla\cdot\mathbf{u}_1 = 0$$

or

$$\frac{\partial}{\partial t}M + \nabla\cdot M\mathbf{u}_1 = 0 \quad , \quad M = \int_{-h}^{0}\overline{\rho}\Big/\int_{-H}^{0}\overline{\rho}$$

yielding

$$g'\nabla h = \left(g'H\frac{\frac{1}{H}\int_{-H}^{0}\overline{\rho}}{\overline{\rho}(-h)}\right)\mathbf{M} = g''M$$

with $H$ the mean thickness of the layer.

Our upper layer equations are

$$\frac{\partial}{\partial t}\mathbf{u}_1 + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u}_1 = -\nabla\left(\phi_2' + g''M + \frac{1}{2}\mathbf{u}_1^2\right)$$
$$\frac{\partial}{\partial t}M + \nabla \cdot M\mathbf{u}_1 = 0$$

For a lower layer with total mass of both layers $\mathcal{M}$

$$\frac{\partial}{\partial t}\mathbf{u}_2 + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u}_2 = -\nabla\left(\phi_2' + \frac{1}{2}\mathbf{u}_2^2\right)$$
$$-\frac{\partial}{\partial t}M + \nabla \cdot (\mathcal{M} - M)\mathbf{u}_2 = 0$$

We will now shift back to the standard notation $(g'', M, \mathcal{M}, \phi_2' \to g', h, H, \phi_2)$.

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}_1 + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u}_1 &= -\nabla\left(\phi_2 + g'h + \frac{1}{2}\mathbf{u}_1^2\right)\\
\frac{\partial}{\partial t}h + \nabla \cdot h\mathbf{u}_1 &= 0\\
\frac{\partial}{\partial t}\mathbf{u}_2 + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u}_2 &= -\nabla\left(\phi_2 + \frac{1}{2}\mathbf{u}_2^2\right)\\
-\frac{\partial}{\partial t}h + \nabla \cdot (H-h)\mathbf{u}_2 &= 0
\end{aligned} \qquad (two-layermodel)$$

The rigid lid and bottom assumption involves neglecting $\frac{\partial}{\partial t}H$; since we've lost a predictive equation, we must treat the pressure $\phi_2$ as a diagnostic variable (i.e., if we multiply the first eqn. by $h$ and the third by $H-h$, and take the divergences, we eliminate the $\frac{\partial}{\partial t}$ to get a Poisson equation for pressure. A similar problem occurs in the anelastic and Boussinesq eqns — the pressure adjusts to that required to maintain non-divergence. If we retain the changes in elevation of the free-surface, we eliminate that problem but then have fast surface gravity waves which can lead to numerical problems.

## 6 —Geostrophic adjustment

One example which illustrates the importance of PV in the slow evolution of eddies and jets is the geostrophic adjustment problem (Rossby, 1938). one-d adjustment two-d adjustment . We consider solving the linear upper layer SW equation (with $\phi_2 = 0$) starting from an initial $\mathbf{u}_h$ and $h = H + \phi/g'$ in vorticity-divergence form. The evolution of the vorticity $\zeta = \nabla^2\psi$ and the divergence $D = -\nabla^2\varphi$

$$\frac{\partial}{\partial t}\zeta + \nabla \cdot (f\mathbf{u}) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\zeta + fD + \beta v = 0$$

$$\frac{\partial}{\partial t}D + \nabla \cdot (\hat{\mathbf{z}} \times f\mathbf{u}) = -\nabla^2\phi \quad \Rightarrow \frac{\partial}{\partial t}D - f\zeta + \beta u = -\nabla^2\phi$$

couples with the mass equation

$$\frac{\partial}{\partial t}\phi + g'HD = 0$$

These are correct for the sphere ($f = 2\Omega\sin\theta$, $\beta = 2\Omega\cos\theta/a$) or the beta plane ($f = f_0 + \beta y$). Eliminating $D$ gives

$$\frac{\partial}{\partial t}Q + \beta v = 0$$

$$Q = \zeta - \frac{f\phi}{g'H}$$

$$-\frac{1}{gH_e}\frac{\partial^2}{\partial t^2}\phi + \nabla^2\phi - \frac{f^2}{g'H}\phi + \beta u = fQ$$

In the absence of $\beta$, this equation has zero-frequency solutions

$$\frac{\partial}{\partial t}Q = 0 \quad , \quad \left(\nabla^2 - \frac{f^2}{g'H}\right)\phi_{par} = fQ$$

These will be geostrophic and non-divergent. The time-dependent part comes from the homogeneous terms

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - g'H\nabla^2\right)\phi_{hom} = 0$$

and represents the gravity waves with characteristic super-inertial frequencies $\omega^2 = f^2 + g'HK^2$. In the presence of $\beta$, the zero-frequency waves become Rossby waves, and we can approximate their solution by neglecting $\varphi$ and replacing the divergence equation $\nabla \cdot f\nabla\psi = \nabla^2\phi$ by $\phi = f\psi$ to give

$$\frac{\partial}{\partial t}Q + \beta\frac{\partial}{\partial x}\psi = 0 \quad , \quad Q = \nabla^2\psi - \frac{f^2}{g'H}\psi$$

with sub-inertial frequencies $\omega = -\beta k/(K^2+\gamma^2)$ with $\gamma^2 = f_0^2/g'H$. The Rossby deformation radius $R_d = 1/\gamma = \sqrt{g'H}/f_0$ gives the length over which the PV anomalies "generate" significant velocities.

Thus we see that the slow modes are associated with PV anomalies and $\beta$ while the fast gravity waves have no PV signal (except, again, for $\beta$-effects). In the nonlinear problem, the slow modes, even on the $f$-plane will evolve by advection of PV. The decomposition is not, however, rigorous: as Ford, et al. (2000) have shown, balanced vortices can be unstable and radiate long gravity waves or generate gravity modes via non-linear interactions between balanced modes (c.f. Remmel and Smith, 2009).

We note that the hydrostatic, fully stratified, linear problem can be separated; in that case the horizontal equation is the same but $g'H \to gH_e$ where the equivalent depth $H_e$ is an eigenvalue of the vertical structure equation

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}F = -\frac{1}{gH_e}F$$

Likewise for the two-layer model, the baroclinic mode has an "equivalent depth" $gH_e = g'H_1H_2/(H_1+H_2)$.

## 7 —Quasi-geostrophic approximation

Traditionally, the QG approximation is done by a Rossby number expansion of the equations. We're going to work from the layer version of Ertel's theorem. Conservation of potential vorticity here implies

$$\frac{D}{Dt}q = 0 \quad , \quad q = \frac{\zeta + f}{h} \tag{7.1}$$

This comes from the vorticity equation

$$\frac{\partial}{\partial t}\zeta_a + \nabla \cdot (\mathbf{u}\zeta_a) = 0 \quad , \quad \zeta_a = f + \zeta$$

or

$$\frac{\partial}{\partial t}qh + \nabla \cdot (\mathbf{u}qh) = 0$$

Combining this with the mass equation leads to the conservation statement (7.1). We define

$$h = \frac{H}{1-(\eta/H)} \quad \text{or} \quad \frac{\eta}{H} = \frac{h-H}{h}$$

Then the conserved property Q=$Hq$ has the form

$$Q = f + \zeta - f\frac{\eta}{H} - \zeta\frac{\eta}{H}$$

For QG, we approximate this by

- dropping the last term (order Rossby number)

$$Q \simeq f + \nabla^2\psi - f\frac{\eta}{H}$$

- noting that the divergent flow is order $Ro$: geostrophy implues $\eta \sim fUL/g'$ so that the terms in the mass equation

$$\frac{1}{h}\frac{D}{Dt}h + \nabla\cdot\mathbf{u}_h = 0 \quad \text{or} \quad -h\frac{D}{Dt}\frac{1}{h} + \nabla\cdot\mathbf{u}_h = 0$$

have sizes

$$\begin{array}{c} \dfrac{1}{H-\eta}\dfrac{D}{Dt}\eta + \nabla\cdot\mathbf{u}_h = 0 \\ fU^2/g'H \qquad \varphi/L^2 \end{array}$$

and $\varphi \sim UL\,Ro\,F$ whereas $\psi \sim UL$. The $\eta/H$ term can be dropped compared to 1 under the choice that $H$ is the mean thickness; deviations are small ($\eta/H \sim RoF$).

- Advection simplifies to

$$\frac{\partial}{\partial t}Q + [\psi, Q] = 0$$

with $[A, B] = \hat{\mathbf{z}}\cdot(\nabla A \times \nabla B)$.

- Finally, we need to relate $\eta$ to the streamfunction; geostrophy again gives

$$g'\eta = f\psi$$

in the single layer case or

$$g'\eta = f(\psi_1 - \psi_2)$$

in the two layer problem.

$$\frac{\partial}{\partial t}Q + [\psi, Q] = 0$$

$$Q = \nabla^2\psi - \frac{f^2}{g'H}\psi + f \tag{QG}$$

or

$$Q_1 = \nabla^2\psi_1 - \frac{f^2}{g'H_1}(\psi_1 - \psi_2) + f \quad , \quad Q_2 = \nabla^2\psi_2 - \frac{f^2}{g'H_2}(\psi_2 - \psi_1) + f$$

with the first equations applying in each layer. We'll use several shorthands: $F_i = f^2/g'H_i$ or $\gamma^2 = f^2(H_1{+}H_2)/g'H_1H_2$, $f^2/gH_1 = \gamma^2/(1{+}\delta)$, $f^2/gH_2 = \delta\gamma^2/(1{+}\delta)$ with $\delta = H_1/H_2$.

We have chosen this approach, because a parallel version can be used in the stratified case (lecture 5) and it illustrates the connections between Ertel PV and QGPV.

*7.1 — Examples*

Many of the processes in these examples are familiar in the barotropic context; however, we shall discuss instead an upper layer problem ($\delta \to 0$)

$$\frac{\partial}{\partial t} q + [\psi, q] = 0 \quad \text{with} \quad q = \nabla^2\psi - \gamma^2\psi + T(\mathbf{x}) \tag{7.2}$$

with $\gamma^2 = f^2/gH_1$. Here $T$ would be $\gamma^2\psi_2 + f - f_0$ but could also represent topography. The assumption is that the deep field is steady (and could be zero). The deformation radius $R_d = 1/\gamma$ and $\gamma \to 0$ for the BTVE.

Consider first dynamics on an "$f$-plane" meaning the scales are small enough that $\beta L/f$ is much less than other parameters (including the desired time scales in the form $1/f\tilde{t}$). Then $T = f_0$ and

$$q \to Q \quad , \quad \frac{\partial}{\partial t} Q + [\psi, Q] = 0 \quad , \quad \mathcal{L}\psi = Q$$

For the BTVE, $\mathcal{L} = \nabla^2$; for (7.2), $\mathcal{L} = \nabla^2 - \gamma^2$; we can work with

$$\psi(\mathbf{x}) = \int d\mathbf{x}' G(\mathbf{x} - \mathbf{x}')Q(\mathbf{x}')$$

and just specialize the Green's function when needed.

$$[BTVE] \quad G = \frac{1}{2\pi}\ln(|\mathbf{x} - \mathbf{x}') \quad , \quad [EBT] \quad G = -\frac{1}{2\pi}K_0(\gamma r)$$

Point vortices: $Q = q_i\delta\big(\mathbf{x} - \mathbf{x}_i(t)\big)$ [summation convention]. For these

$$\psi(\mathbf{x}) = q_i G(\mathbf{x} - \mathbf{x}_i')$$

The dynamics becomes

$$-q_i\frac{\partial \mathbf{x}_i}{\partial t} \cdot \nabla\delta(\mathbf{x} - \mathbf{x}_i) + q_i\mathbf{u} \cdot \nabla\delta(\mathbf{x} - \mathbf{x}_i) = 0$$

implying

$$\frac{\partial}{\partial t}\mathbf{x}_i = \mathbf{u}(\mathbf{x}_i) = \hat{\mathbf{z}} \times \nabla\psi\big|_{\mathbf{x}_i} = q_j\hat{\mathbf{z}} \times \nabla_i G(\mathbf{x}_i - \mathbf{x}_j)$$

bt exchange merger ebt ebt 1.7 ebt merger

In this calculation, the self-advection term from $j = i$ is dropped. We can justify this by considering desingularized vortices (with compact support) $Q = \sum Q_i(\mathbf{x})$. We assume the vortices are small and widely-separated so we can define an area $D_i$ covering each vortex. Then

$$\frac{\partial}{\partial t}\int_{D_i} Q_i = -\oint_{\partial D_i} \mathbf{u} \cdot \hat{\mathbf{n}}\, Q_i = 0$$

If we now examine the center of mass of the $i^{th}$ vortex

$$X_i = \frac{\int_{D_i} x Q_i}{\int_{D_i} Q_i}$$

we have

$$\begin{aligned}\left(\int_{D_i} Q_i\right)\frac{\partial}{\partial t}X_i &= -\int_{D_i} x[Q,\psi] \\ &= -\oint_{\partial D_i} x\mathbf{u}\cdot\hat{\mathbf{n}}Q + \int_{D_i} uQ \\ &= \int_{D_i} uQ_i\end{aligned}$$

where we have chosen the boundary of $D_i$ to be outside the patch. Likewise

$$\left(\int_{D_i} Q_i\right)\frac{\partial}{\partial t}Y_i = \int_{D_i} vQ_i$$

But the part of $u$ and $v$ deriving from $Q_i$ (i.e., from $\psi_i = \int GQ_i$) will have disappear from the advection terms; e.g.

$$\int_{D_i} v_i Q_i = \iint Q_i(\mathbf{x})\frac{\partial}{\partial x}G(\mathbf{x}-\mathbf{x}')Q_i(\mathbf{x}')$$

will be zero, since the derivative makes the kernel antisymmetric. If the other vortices are separated sufficiently,

$$\psi_j = \int G(\mathbf{x}-\mathbf{x}')Q_j(\mathbf{x}') \simeq G(\mathbf{x}-\mathbf{X}_j)\int Q_j$$

and the patch motion agrees with the point vortex model. We will consider the corrections associated with $f(y)$ later.

PATCH OF VORTICITY: for a nearly circular patch with area $\pi b^2$

$$Q = q_0\mathcal{H}(b+\eta-r)$$

For $\eta = 0$ and $\mathcal{L} = \nabla^2$, we have the Rankine vortex with

$$v(r) = \frac{\partial\psi}{\partial r} = \begin{cases}\frac{1}{2}q_0 r & r < b \\ \frac{1}{2}q_0\frac{b^2}{r} & r > b\end{cases}$$

bt ellipse ebt precession strain weak strain counter shear shear steady

We can understand how the perturbed vortex will evolve by noting that the boundary $r = b + \eta$ is a material surface

$$\frac{\partial}{\partial t}\eta = -\frac{1}{b+\eta}\frac{d}{d\theta}\psi(b+\eta(\theta,t),\theta)$$

We can think in terms of anomalies associated with the contour displacements $\eta$ and how they, in conjunction with the flow around the vortex, move the anomalies. Let

$$\mathcal{L}\overline{\psi} = q_0\mathcal{H}(b-r) \quad , \quad \mathcal{L}\psi' = q_0\mathcal{H}(b+\eta-r) - q_0\mathcal{H}(b-r)$$

with the latter linearizing to

$$\mathcal{L}\psi' = q_0\eta\,\delta(b-r)$$

We also linearize the dynamical equation

$$\frac{\partial}{\partial t}\eta = -\frac{1}{b}\frac{\partial\overline{\psi}}{\partial r}\frac{\partial\eta}{\partial\theta} - \frac{1}{b}\frac{\partial\psi'}{\partial\theta} = -\frac{\overline{v}(b)}{b}\frac{\partial\eta}{\partial\theta} - \frac{1}{b}\frac{\partial\psi'}{\partial\theta}$$

For steadily propagating vortex waves, $\eta(\theta,t) = \eta(\theta - ct)$ and

$$\frac{\overline{v}(b)}{b}\eta + \frac{1}{b}\psi' = c\eta$$

This will happen if $\eta$ is a pure mode $\eta = \eta_0\exp(\imath n\theta)$; then

$$\psi' = \psi'(r)e^{\imath n\theta} \quad , \quad \psi'(r) = q_0 G_n(r|b)\eta_0$$

with $G_n$ the Green's function associated with $\mathcal{L}_n$ – the radial part of $\mathcal{L}$ when $\frac{\partial}{\partial\theta} \to \imath n$. The waves propagate at

$$c = \frac{\overline{v}(b)}{b} + \frac{q_0}{b}G_n(b|b)$$

Taking an $r$ derivative of the $\overline{\psi}$ equation gives

$$\mathcal{L}_1\overline{v} = -q_0\delta(b-r) \quad \Rightarrow \overline{v}(b) = -q_0 G_1(b|b)$$

and

$$c = \frac{q_0}{b}\Big(G_n(b|b) - G_1(b|b)\Big)$$

The Green's function takes the form

$$G(r|r') = \frac{1}{W(G_+,G_-)}\begin{cases} G_-(r)G_+(r') & r < r' \\ G_-(r')G_+(r) & r' < r \end{cases}$$

with the two solutions of $\mathcal{L}_n G_\pm = 0$ satisfying regularity at the origin for $G_-$ and at infinity for $G_+$. For the BTVE,

$$G_n(r|r') = -\frac{r'}{2n}\begin{cases} \left(\frac{r}{r'}\right)^n & r < r' \\ \left(\frac{r'}{r}\right)^n & r > r' \end{cases} \qquad (G - BT)$$

and

$$\begin{aligned} c &= q_0[\frac{1}{2} - \frac{1}{2n}] \\ &= \frac{\overline{v}(b)}{b}[1 - \frac{1}{n}] \end{aligned}$$

The $n = 1$ mode does not precess; it simply represents a displacement of the vortex. The higher modes precess, but at a rate slower than the vortex itself rotates at the boundary. Indeed, the first term gives a precession at the rate of the particles on the boundary, while the second term represents a retrograde propagation from the wave effects, a propagation which weakens as the waves become more wiggly.

For the "equivalent barotropic" (EBT) case with $\gamma \neq 0$,

$$G_n(r|r') = -r'\begin{cases} I_n(\gamma r)K_n(\gamma r') & r < r' \\ I_n(\gamma r')K_n(\gamma r) & r > r' \end{cases} \qquad (G - EBT)$$

and

$$\begin{aligned} c &= q_0\Big(I_1(\gamma b)K_1(\gamma b) - I_n(\gamma b)K_n(\gamma b)\Big) \\ &= \frac{\overline{v}(b)}{b}\left(1 - \frac{I_n K_n}{I_1 K_1}\right) \end{aligned}$$

Again, the $n = 1$ mode simply represents a displacement of the vortex and the higher modes precess, but at a rate slower than the vortex itself rotates at the boundary.

Planetary waves: if we let $\nabla^2\psi = -K^2\psi$, and we define $Q = q - f = -(K^2 + \gamma^2)\psi$ then

$$[\psi, q] = [\psi, -(K^2 + \gamma^2)\psi + f] = [\psi, f] = -\frac{1}{K^2 + \gamma^2}[Q, f]$$

so that

$$\frac{\partial}{\partial t}Q - \frac{1}{K^2 + \gamma^2}[Q, f] = 0$$

On the sphere (with $\varphi$ the longitude)

$$Q = Q(\varphi + \frac{2\Omega}{a^2(K^2 + \gamma^2)}t)$$

— the crests move westward with an angular speed $c = -2\Omega/a^2(K^2 + \gamma^2)$ proportional to their "length scale" squared. For the EBT system, unlike the BTVE, this is an approximate statement, since we used the QG equation which is problematic at the equator. We'll return to that in a moment.

For the BTVE, the wave structures are

$$\psi = P_n^m(\cos\theta)\cos m\varphi$$

where the $P_n^m$ are Legendre functions and

$$K^2 a^2 = n(n+1)$$

Examples are

$$\begin{aligned} \psi &= \cos\theta\cos\varphi \quad , \quad c = -\Omega \\ \psi &= \sin 2\theta\cos\varphi \quad , \quad c = -\Omega/3 \\ \psi &= \cos^2\theta\cos 2\varphi \quad , \quad c = -\Omega/3 \end{aligned}$$

Beta-plane waves: again, we have $\nabla^2\psi = -K^2\psi$ but now find plane waves

$$\psi = \cos(kx + \ell y - kct) \quad , \quad k^2 + \ell^2 = K^2$$

(although there are other solutions such as $J_0(Kr)$). These move westward at

$$c = -\beta/(K^2 + \gamma^2)$$

Equatorial beta-plane waves: if we make a conformal transformation of the sphere to a mercator map, the EBT shallow water equations become

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u}^* + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u}^* &= -\nabla B \\ \frac{\partial}{\partial t}h + S\nabla\cdot\mathbf{u}^* h &= 0 \\ \zeta &= S\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{u}^*) \\ B &= gh + S\frac{1}{2}|\mathbf{u}^*|^2 \end{aligned}$$

with $\mathbf{u}^* = s\mathbf{u}$, $S = 1/s^2$, and $s$ the scale factor $s = \cos\theta = \mathrm{sech}(y/a)$. $f = 2\Omega\sin\theta = 2\Omega\tanh(y/a)$. The operators are now all in cartesian form $\left(\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)\right)$. Numerically, these are no more difficult to solve than the standard SW equations (except for CFL conditions). The equatorial beta-plane equations simplify these by keeping order 1 and $y/a$ but dropping higher order powers of $y/a$

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u} + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u} &= -\nabla B \\ \frac{\partial}{\partial t}h + \nabla\cdot\mathbf{u}h &= 0 \\ \zeta &= \hat{\mathbf{z}}\cdot(\nabla\times\mathbf{u}) \\ B &= gh + \frac{1}{2}|\mathbf{u}^*|^2 \end{aligned}$$

with $f = \beta y$ and $\beta = 2\Omega/a$. These equations have an exact linear dispersion relation deriveable from

$$\left(\nabla^2 - \frac{f^2}{g'H}\right)\frac{\partial}{\partial t}v + \beta\frac{\partial}{\partial x}v - \frac{1}{gH}\frac{\partial^3}{\partial t^3}v = 0 \tag{7.3}$$

For

$$v = H_n(\frac{y}{R})\exp(-\frac{1}{2}\frac{y^2}{R^2})$$

where the $H_n$ are the Hermite polynomials, we end up with

$$\frac{\omega^2}{gH} - \frac{\beta k}{\omega} - k^2 = \frac{1}{R^2}(2n+1) \tag{7.4}$$

and

$$R^4 = gH/\beta^2$$

The quantity

$$R = (gH)^{1/4}\beta^{-1/2}$$

is called the "equatorial deformation radius" and is the fundamental disturbance scale in the north-south direction.

The cubic has three roots: two gravity waves with

$$\omega^2 \simeq gH(k^2 + \frac{2n+1}{R^2})$$

and a Rossby mode with

$$\omega \simeq -\frac{\beta k}{k^2 + \frac{2n+1}{R^2}} \tag{7.5}$$

If we're only interested in the latter, we can drop the last term in (7.3); the resulting equation is the same as the linearized *(QG)* equation if we retain $f = \beta y$. Therefore the wave propagation characteristics match (7.5). We lose the Kelvin wave and the high frequenct (gravity wave-like) limit of the $n = 0$ Yanai wave, but retain the low-frequency motions.

We shall discuss the influence of $\beta$ on vortices and jets later.

Stability of a vortex

We shall go into detail on this later also; here, let's just talk about the physics. Suppose we have

$$Q = \mathcal{H}(1 + \eta_1 - r) + q_2\mathcal{H}(b + \eta_2 - r)$$

but with $q_2$ being negative. (Lengths have been normalized by the inner radius and times by the inverse of the vorticity jump from inside to outside this radius. For convenience, consider an isolated vortex with no net circulation so that $q_2 = -1/b^2$ and the maximum swirl velocity is at $r = 1$. If we perturb the inner interface, the waves will tend to propagate clockwise but be advected counter-clockwise. Waves on the outer boundary will propagate counter-clockwise; for appropriate scales, it may be possible for these to be nearly in sync. But the waves on the inner interface can amplify those on the outer and vice-versa. If they stay in the right phase relationship with each other, the perturbations will grow.

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