Fluid Dynamics in Earth and Planetary Sciences
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Lecture 5: Eddies and jets: Jovian dynamics

- Standard QG models
  - Eddies in shear
- 1.5 layer model
- 1.75 layer model
  - Steady vortices in shear
- Two-beta model
- Deep QG

full
band
red spot
planet
spot
false color
winds I
around spot DI
streamlines

1 —Standard QG models

1.1 — Eddies in shear

We will show that an isolated vortex has a smaller vertical scale than the jets. If

$$q_i = \overline{q}_i(y) + q_i'(\mathbf{x})$$

with $\overline{q}_i = A\cos(\ell y)$ (deep shear flow), then, on all streamlines extending to infinity

$$\overline{\psi}_i + \psi_i' = -\frac{1}{\ell^2}(\overline{q}_i + q_i')$$

so that

$$\nabla^2 \psi_i' \pm F_i(\psi_2' - \psi_1') = -\ell^2 \psi_i'$$

From this, we find the equations for the barotropic and baroclinic components

$$\left(\nabla^2 + \ell^2\right)\left(H_1\psi_1' + H_2\psi_2'\right) = 0$$
$$\left(\nabla^2 - F_1 - F_2 + \ell^2\right)\left(\psi_1' - \psi_2'\right) = 0$$

But for an isolated solution, which decays like exponential or K_n Bessel functions, we will have $\nabla^2\psi \sim \psi$. This is inconsistent with the vortex having an exterior barotropic component, but can occur for the baroclinic problem if

$$\ell^2 < F_1 + F_2 = \frac{1}{R_d^2}$$

The length scale of the jets must be larger than the deformation radius.

2 —1.75 layer model

centered
sa
no deep flow
infinite layer
deep jets, standard atmospheric model
2-beta

If we have an infinitely deep lower layer but with steady jet flows like the upper layer,

$$U_1 = U_2 = U\sin(\ell y)$$

the “topographic” PV is just $T = F\psi_2 = (FU/\ell)\cos(\ell y)$. Our far-field flow is $\Psi = (U/\ell)\cos(\ell y)$, and the PV is $-\ell U\cos(\ell y) - (FU/\ell)\cos(\ell y) + T = -\ell U\cos(\ell y) = -\ell^2\Psi$. Our conserved invariant is

$$A = -\frac{1}{2}\iint \big(q(\mathbf{x}) - T(\mathbf{x})\big)G(\mathbf{x}-\mathbf{x}')\big(q(\mathbf{x}') - T(\mathbf{x}')\big) - \frac{1}{2}\frac{1}{\ell^2}\int q^2$$

or, writing $q = \overline{q} + q'$ where $\overline{q} = -\ell^2\Psi$,

$$\begin{aligned}
A - \overline{A} &= -\iint q'(\mathbf{x})G(\mathbf{x}-\mathbf{x}')\big(\overline{q}(\mathbf{x}') - T(\mathbf{x}')\big) - \frac{1}{\ell^2}\int \overline{q}q' - \frac{1}{2}\iint q'(\mathbf{x})G(\mathbf{x}-\mathbf{x}')q'(\mathbf{x}') - \frac{1}{2}\frac{1}{\ell^2}\int q'^2 \\
&= -\int q'(\mathbf{x})\Psi(\mathbf{x}) + \int \Psi q' - \frac{1}{2}\iint q'(\mathbf{x})G(\mathbf{x}-\mathbf{x}')q'(\mathbf{x}') - \frac{1}{2}\frac{1}{\ell^2}\int q'^2 \\
&= -\frac{1}{2}\iint q'(\mathbf{x})G(\mathbf{x}-\mathbf{x}')q'(\mathbf{x}') - \frac{1}{2}\frac{1}{\ell^2}\int q'^2
\end{aligned}$$

The first term is positive definite and the second negative definite. The flow will be stable if

$$\int q'^2 > -\ell^2\int \psi' q'$$

If we Fourier-transform q', this implies

$$\int d\mathbf{k}|\hat{q}|^2 > \int d\mathbf{k}\frac{\ell^2}{|\mathbf{k}|^2 + F}|\hat{q}|^2$$

This will be true if $\ell^2 < F$, giving again the condition that the deformation radius is less than the jet scale.

2.1 — Steady vortices in shear

The equations here

$$\frac{\partial}{\partial t}q = [q, \psi] \quad , \quad q = (\nabla^2 - F)\psi + T \quad , \quad T = F\psi_2 = F\frac{U}{\ell}\cos(\ell y)$$

will have steady solutions if

$$q = \mathcal{Q}(\psi)$$

We choose a simple representation

$$q = -\ell^2\psi + q_0\mathcal{H}(b + \eta - r) \quad \Rightarrow \quad \nabla^2\psi - \alpha^2\psi + F\frac{U}{\ell}\cos(\ell y) = q_0\mathcal{H}(b + \eta - r)$$

with $\alpha^2 = F - \ell^2$; the curve $b + \eta$ must be a streamline. This will have isolated solutions if $F > \ell^2$, again requiring the deformation radius to be smaller than the jet scale. The solutions can be written as

$$\psi = \frac{U}{\ell}\cos(\ell y) + \psi' \quad , \quad \nabla^2\psi' - \alpha^2\psi' = q_0\mathcal{H}(b + \eta - r) \quad , \quad \psi(b + \eta(\theta), \theta) = \psi_0$$

If the vortex is small and only weakly perturbed, the $\cos 2\theta$ part of ψ' is just

$$\psi_2' = q_0 G_2(r|b)\eta_2$$

and

$$\frac{\partial}{\partial r}\psi_0' = -g_0 G_1(r|b)$$

The estimated value of ψ' on the boundary is therefore

$$\psi'(b + \eta, \theta) \simeq q_0\big(G_2(b|b) - G_1(b|b)\big)\eta_2 \cos 2\theta$$

This must cancel the η_2 part of $(U/\ell)\cos(\ell b \sin\theta)$. Projecting that out gives $(U/\ell)2J_2(\ell b)$. (In the limit of small b, that's just $U\ell b^2/4$.) We could include the $\cos(4\theta)$, $\cos(6\theta)$ ... modes as well. But, here we have

$$\eta_2 = \frac{2UJ_2(\ell b)}{q_0(G_1 - G_2)}$$

with $G_n(b|b) = -bI_n(\alpha b)K_n(\alpha b)$. Of course, this breaks down when η becomes too large: the boundary becomes indented at the north and south. (This happens when $\eta > 0.2b$.)

We can find larger amplitude solutions by a simple root-finding procedure:

- given a set of points on the estimated boundary, use contour dynamics to find the velocities for ψ' in between the points
- add the $U\sin(\ell y)\hat{\mathbf{x}}$ contributions and calculate the normal components
- move the points until these normal velocities vanish.

steady vortices in shear b=0.5 b=1.4 std sa
dy=1 std sa dbsa
other offsets pi/6 pi/2

We can also use the simulated annealing procedure to find more extreme states in the continuous model.

3 —Deep QG

Let us begin considering some puzzles about the Red Spot and the jets. The jets appear to have reversals in $\frac{\partial}{\partial y}\overline{q}$ yet are stable over decades. (There are small-scale eddies, but no large-scale roll-up). The spots are long-lived, though some of the others have disappeared, and the Red Spot is apparently weakening, though we don't know if that is transient or not. But, despite the turbulence on its edges, it has certainly persisted.

From the vorticity equation, we find the component along some direction $\hat{\mathbf{n}}$ satisfies

$$\frac{\partial}{\partial t}Z + \nabla \cdot Z\mathbf{u} = (2\mathbf{\Omega} + \boldsymbol{\zeta}) \cdot \nabla(\hat{\mathbf{n}} \cdot \mathbf{u}) + \hat{\mathbf{n}} \cdot (\nabla \times g\tau\hat{\mathbf{r}})$$

with $Z = \hat{\mathbf{n}} \cdot (2\mathbf{\Omega} + \boldsymbol{\zeta})$. The last term is equal to $\nabla g\tau \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{n}})$. Expanding $\nabla \cdot (Z\mathbf{u})$ and using the mass equation gives

$$\frac{\partial}{\partial t}Z + \mathbf{u} \cdot \nabla Z - Z\mathbf{u} \cdot \nabla \ln \overline{\rho} = (2\mathbf{\Omega} + \boldsymbol{\zeta}) \cdot \nabla(\hat{\mathbf{n}} \cdot \mathbf{u}) + \nabla g\tau \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{n}})$$

Like Ingersoll and Pollard(1982) and Yano and Flierl (1994), we now take $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ (with the case $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ dealt with below); then $Z = \zeta + 2\Omega$ and the small Rossby number version is

$$\frac{\partial}{\partial t}\zeta + \mathbf{u} \cdot \nabla\zeta - 2\Omega\mathbf{u} \cdot \nabla \ln \overline{\rho} = 2\Omega\frac{\partial}{\partial z}(\hat{\mathbf{z}} \cdot \mathbf{u}) + \nabla g\tau \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{z}})$$

or

$$\frac{D}{Dt}(\zeta - 2\Omega \ln \overline{\rho}) = 2\Omega\frac{\partial}{\partial z}(\hat{\mathbf{z}} \cdot \mathbf{u}) + \nabla g\tau \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{z}})$$

The anelastic density structure takes the place of the β-effect. As we shall see, it is negative. Writing the velocity in terms of the components perpendicular and parallel to $2\mathbf{\Omega}$, $\mathbf{u} = \mathbf{u}_\perp + w\hat{\mathbf{z}}$ gives

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla\right)(\zeta - 2\Omega \ln \overline{\rho}) + w\frac{\partial}{\partial z}\zeta = \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w + \nabla g\tau \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{z}})$$

The thermal wind balance

$$2\Omega\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi + g\tau\hat{\mathbf{r}}$$

gives

$$\frac{\partial}{\partial z}\phi = g\tau\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$$

and, for the velocities perpendicular to $\mathbf{\Omega}$,

$$2\Omega\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla\phi - g\tau\hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

When the fluid is isentropic ($\tau = 0$), the pressures are independent of depth and the horizontal velocities are also – $\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla(\phi/2\Omega)$. This implies $\zeta = \nabla^2\psi$ with $\psi = \phi/\Omega$

acting like a streamfunction for the perpendicular flow. $w\frac{\partial}{\partial z}\zeta$ is negligible and we can multiply the vorticity equation by $\overline{\rho}$ and integrate; if the surface density goes to zero

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla\right) q = 0 \quad , \quad q = \zeta \int dz \overline{\rho} - 2\Omega \int dz \overline{\rho} \ln \overline{\rho}$$

Since ζ is small compared to 2Ω, changes in ζ as a column of fluid moves north will be dominated by the second term, proving a term

$$-2\Omega \int dz\, \overline{\rho} \ln \overline{\rho} \Big/ \int dz\, \overline{\rho}$$

analogous to the beta-effect but decreasing with latitude.

An alternate derivation begins with the anelastic equations, splitting the buoyancy field into a background stratification depending on radius plus deviations

$$\tau = \overline{\tau}(r) + \tau'$$

Ertel's theorem then takes the form

$$\begin{aligned} q &= \frac{1}{\overline{\rho}} \left(\nabla\overline{\tau} \cdot (2\mathbf{\Omega} + \boldsymbol{\zeta}) + \nabla\tau' \cdot (2\mathbf{\Omega} + \boldsymbol{\zeta})\right) \\ &= \frac{\overline{\tau}_r}{\overline{\rho}} \left(f + \zeta + \frac{1}{\overline{\tau}_r} 2\mathbf{\Omega} \cdot \nabla\tau' + \frac{1}{\overline{\tau}_r} \boldsymbol{\zeta} \cdot \nabla\tau'\right) \\ &\equiv \frac{\overline{\tau}_r}{\overline{\rho}} q' \end{aligned}$$

The conservation equation gives

$$\frac{D}{Dt} q' + q' \frac{\overline{\rho}}{\overline{\tau}_r} \frac{D}{Dt} \frac{\overline{\tau}_r}{\overline{\rho}} = 0 \quad \text{or} \quad \frac{D}{Dt} q' - q' \frac{\overline{\tau}_r}{\overline{\rho}} \frac{D}{Dt} \frac{\overline{\rho}}{\overline{\tau}_r} = 0$$

Using the fact that $\overline{\rho}$ and $\overline{\tau}$ depend only on r and the conservation of $\overline{\tau} + \tau'$ turns this into

$$\frac{D}{Dt} q' - q' \frac{\overline{\tau}_r}{\overline{\rho}} w \frac{\partial}{\partial r} \left(\frac{\overline{\rho}}{\overline{\tau}_r}\right) = 0 \quad , \quad \frac{D}{Dt} q' + q' \frac{1}{\overline{\rho}} \frac{\partial}{\partial r} \left(\frac{\overline{\rho}}{\overline{\tau}_r}\right) \frac{D}{Dt} \tau' = 0$$

This is still an exact statement; now, however, we will drop higher order terms in the Rossby number. This gives

$$\frac{D}{Dt} \left(f + \zeta + \frac{1}{\overline{\tau}_r} 2\mathbf{\Omega} \cdot \nabla\tau'\right) + f \frac{1}{\overline{\rho}} \frac{\partial}{\partial r} \left(\frac{\overline{\rho}}{\overline{\tau}_r}\right) \frac{D}{Dt} \tau' = 0 \quad , \quad \frac{D}{Dt} c = \frac{d}{dt} c + \hat{\mathbf{r}} \left(\nabla\psi \times \nabla c\right)$$

where we also neglect advection by the divergent flow. We also make the two-scale approximation $\frac{\partial}{\partial \theta} \to \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial \Theta}$. All the derivatives on the dynamical fields are on the short scale, so that $f = 2\Omega \sin\Theta$ is constant and the only Θ derivative that appears in on f, leading to the β-effect term. The two-scale formalism implies that we can pull the coefficient inside the $\frac{D}{Dt}\tau'$ to get an approximately conserved property

$$\frac{D}{Dt}\left(f + \zeta + \frac{1}{\overline{\tau}_r}2\mathbf{\Omega}\cdot\nabla\tau' + f\tau'\frac{1}{\overline{\rho}}\frac{\partial}{\partial r}\left(\frac{\overline{\rho}}{\overline{\tau}_r}\right)\right) = 0$$

or

$$\frac{D}{Dt}(Q + f) = 0 \quad , \quad Q = \nabla_h^2\psi + \frac{1}{\overline{\rho}}2\mathbf{\Omega}\cdot\nabla\left(\frac{\overline{\rho}}{\overline{\tau}_r}\tau'\right)$$

Finally, we need to relate buoyancy perturbations τ' to ψ. To do this, we use the generalized thermal wind equation

$$2\mathbf{\Omega}\times(\hat{\mathbf{r}}\times\nabla\psi) = \hat{\mathbf{r}}\;2\mathbf{\Omega}\cdot\nabla\psi - f\nabla\psi = \nabla\phi + \tau'\nabla\Phi$$

Taking the curl (remembering that f is constant under the two-scale approximation) gives

$$-\hat{\mathbf{r}}\times\nabla(2\mathbf{\Omega}\cdot\nabla\psi) = g\nabla\tau'\times\hat{\mathbf{r}}$$

so that

$$g\tau' = 2\mathbf{\Omega}\cdot\nabla\psi$$

and our QGPV becomes

$$\boxed{\frac{D}{Dt}(Q + f) = 0 \quad , \quad Q = \nabla_h^2\psi + \frac{1}{\overline{\rho}}2\mathbf{\Omega}\cdot\nabla\left(\frac{\overline{\rho}}{N^2}2\mathbf{\Omega}\cdot\nabla\psi\right) \qquad (\textit{Deep QG})}$$

with $N^2 = g\overline{\tau}_r$. For a shallow fluid, $2\mathbf{\Omega}\cdot\nabla$ becomes $f\frac{\partial}{\partial z}$ and we recover the standard QG equations, but for a deep, weekly stratified interior, we see a tendency towards Taylor columns $2\mathbf{\Omega}\cdot\nabla\psi \to 0$.

4 —Two-beta model

sketch

The previous derivations argue that the deep fluid has a negative and strong β-effect. Incorporating this in the QG two-layer model gives

$$\begin{aligned}
\frac{\partial}{\partial t}q_i &= [q_i, \psi_i] \\
q_1 &= \nabla^2\psi_1 - F_1(\psi_1 - \psi_2) + \beta_1 y \\
q_2 &= \nabla^2\psi_2 - F_2(\psi_2 - \psi_1) + \beta_2 y
\end{aligned}$$

4.1 — Baroclinic instability

With opposite-signed β, the Rayleigh criterion for baroclinic instability is satisfied even without shear. Thus instability occurs for very small shears and thermal gradients. The stability problem is

$$\left(\mathbf{U} + \mathbf{Q_y}\mathbf{L}^{-1}\right)\mathbf{q}' = c\mathbf{q}'$$

with

$$\mathbf{q}' = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \quad , \quad \mathbf{Q_y} = \begin{pmatrix} Q_{1y} & 0 \\ 0 & Q_{2y} \end{pmatrix}$$

and

$$\mathbf{L} = \begin{pmatrix} -K^2 - F_1 & F_1 \\ F_2 & -K^2 - F_2 \end{pmatrix}$$

The PV gradients are

$$Q_y = \begin{cases} \beta_1 - \frac{\partial^2}{\partial y^2}U_1 + F_1(U_1 - U_2) \\ \beta_2 - \frac{\partial^2}{\partial y^2}U_2 - F_2(U_1 - U_2) \end{cases}$$

Consider pure BCI without the jets. For $\beta_1 > 0$ and $\beta_2 < 0$, the Rayleigh criterion for instability will be satisfied even for $U_1 = U_2 = 0$ as mentioned above. But the Fjortoft criterion for stability requires

$$UQ_y$$

to be negative for both layers. That will happen if

$$\beta_2/F_2 < U_1 - U_2 < -\beta/F_1$$

Eastward shears (expected from solar heating) will be unstable for arbitrarily small values but will, concomitantly, have small growth rates.

bci

jet generation

jet structure

deep jets

In the two-beta model,

- for small shears, the instabilities are weak and small scale
- this can force jets which are quite zonal
- $\beta - U_{yy}$ can change sign in the upper layer

References

[1] Ingersoll, A.P. and Pollard, D., 1982: Motion in the interiors and atmospheres of Jupiter and Saturn: scale analysis, anelastic equations, barotropic stability criterion, Icarus, 52, 62-80. (p4, Ingersoll and Pollard, 1982)

[2] Yano, J.-I. and Flierl, G.R., 1994: Jupiter's great red spot: Compactness condition and stability, Ann. Geophys., 12, 1-18. (p4, Yano and Flierl, 1994)