A Linear Stability Analysis on the Onset of Thermal Convection of a Fluid with Strongly Temperature-dependent Viscosity in a Spherical Shell

Masanori Kameyama

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Acknowledgement

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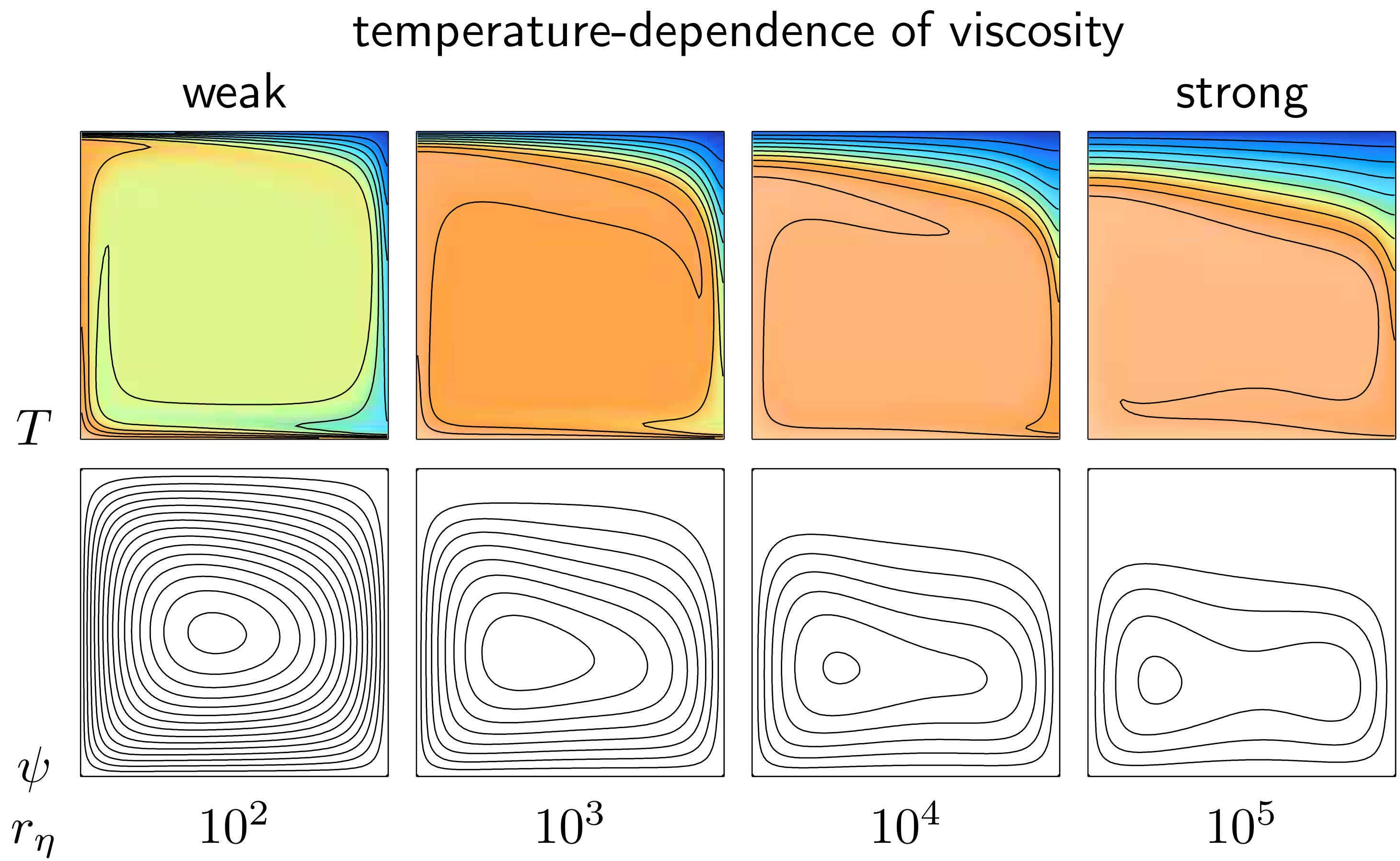
Introduction

Changes in vertical flow structures due to $\eta(T)$

(Solomatov, 1995)



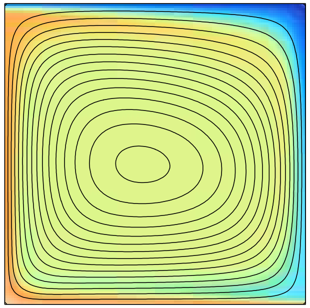
Note: The viscosity of the hottest fluid η_b is kept constant, yielding the Rayleigh number of $Ra_b = 10^7$ (where Ra_b is defined with η_b).

Transitions in convective regimes (2)

Changes in vertical flow structures (Solomatov, 1995)

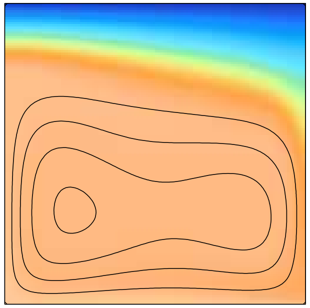
Three convective regimes are identified depending on the temperature-dependence of viscosity:

- "Whole-Layer" (WH) mode (for weak dependence) thin cold thermal boundary layer is involved into convection at depth.



- Transitional mode (for moderate dependence)

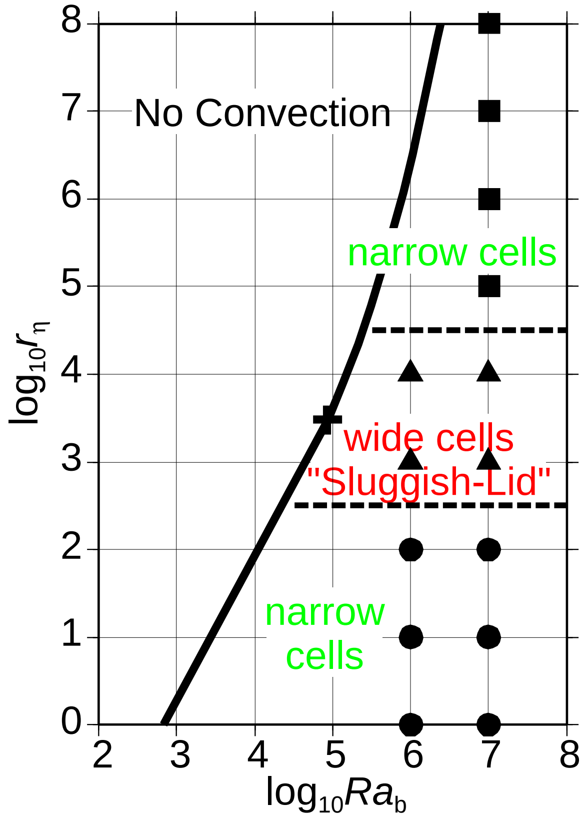
- "Stagnant-Lid" (ST) mode (for strong dependence) thick and stiff cold thermal boundary layer (or "lid") is NOT involved into convection at depth.

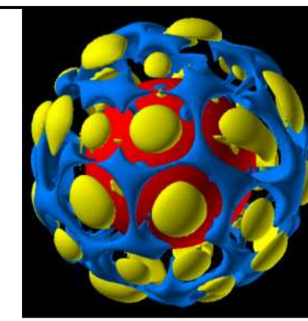
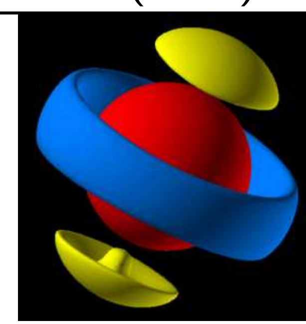
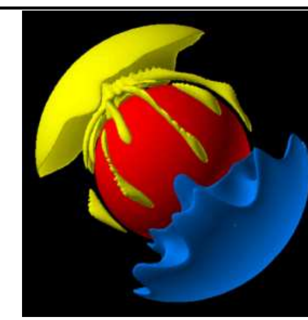


Transitions in convective regimes (3)

Changes in horizontal flow structures due to $\eta(T)$

(figures from Yoshida and Kageyama, 2005)



r_η	Ra_b 10^6	Ra_b 10^7
10^6	thermal anomalies hot (yellow) cold (blue)	
10^4		

Horizontal length scale of convection changes from narrow to wide, and then from wide to narrow ones with increasing temperature-dependence of viscosity.

Questions to be addressed in this study

➢Acknowledgement

Introduction

➢transitions (1)

➢transitions (2)

➢transitions (3)

➢questions

➢what we do

Numerical Model

Result 1: isoviscous

Results 2 : $\eta(T)$ in a planar layer

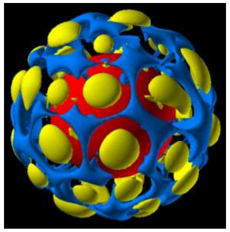
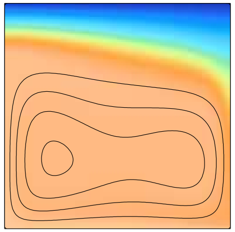
Results 3 : $\eta(T)$ in a spherical shell

Discussion and Concluding Remarks

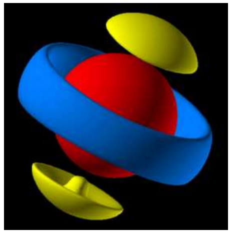
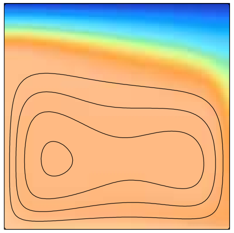
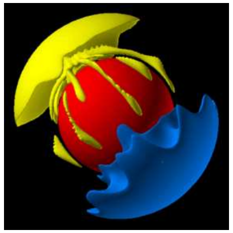
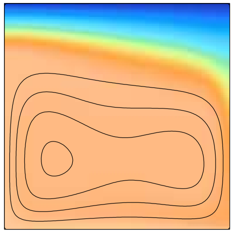
Linear Stability Analysis

analytically for shell

- ❑ What is the relation between the changes in vertical and horizontal flow structures ?
- ❑ How convection looks when viscosity is strongly temperature-dependent ?
  Is it always characterized by convection cells of small horizontal length scales beneath cold stiff lids ?

 $\equiv$  ?

- ❑ If not, under which conditions does convection simultaneously have cells of large horizontal length scales and cold stiff lids ?

 $\in$  ? and/or  $\in$  ?

What we do in this study

Linear stability analysis on the onset of thermal convection of a fluid with strongly temperature-dependent viscosity in a spherical shell geometry.

- ❏ to classify the flow patterns of spherical shell convection, from the changes in the vertical flow structures,

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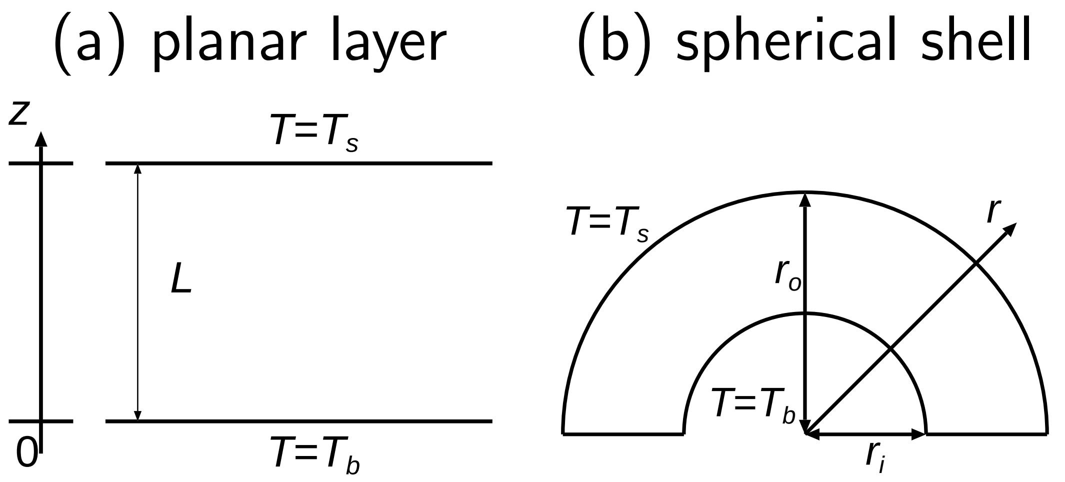
1. onset of convection in a planar layer
   - ❏ to develop a criterion for transition into ST regime
2. onset of convection in a spherical shell
   - ❏ to identify the transition into ST regime by using same criterion

Numerical Model

Geometry



$$\gamma \equiv r_i/r_o (< 1)$$

- Onset of thermal convection driven by basal heating (i.e., no internal heat sources)
- Exponential temperature-dependence of viscosity

$$\eta \propto \exp\left(-E\frac{T-T_s}{T_b-T_s}\right)$$

- top/bottom boundaries are free-slip (F) or rigid (R)

Basic Equations (dimensionless)

❑ Equation of heat transport

$$\frac{\partial T}{\partial t} + \underbrace{\boldsymbol{v} \cdot \nabla T}_{\text{advection}} = \underbrace{\nabla^2 T}_{\text{conduction}}$$

❑ Equation of continuity (incompressible fluid)

$$\nabla \cdot \boldsymbol{v} = 0$$

❑ Equations of motion (force balance)

$$0 = -\nabla p + \nabla \cdot \left[\eta \left(\nabla \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \nabla\right)\right] + RaT\boldsymbol{e}_g$$

❑ Constitutive Equation (temperature-dependence of viscosity)

$$\eta = \eta_{1/2} \exp\left[-E\left(T - \tfrac{1}{2}\right)\right]$$

where Ra : Rayleigh number (defined with $\eta_{1/2}$, not with η_{b})
E : temperature-dependence of viscosity

In this study, linearized equations are solved for infinitesimal perturbations in T and $\boldsymbol{v}$.

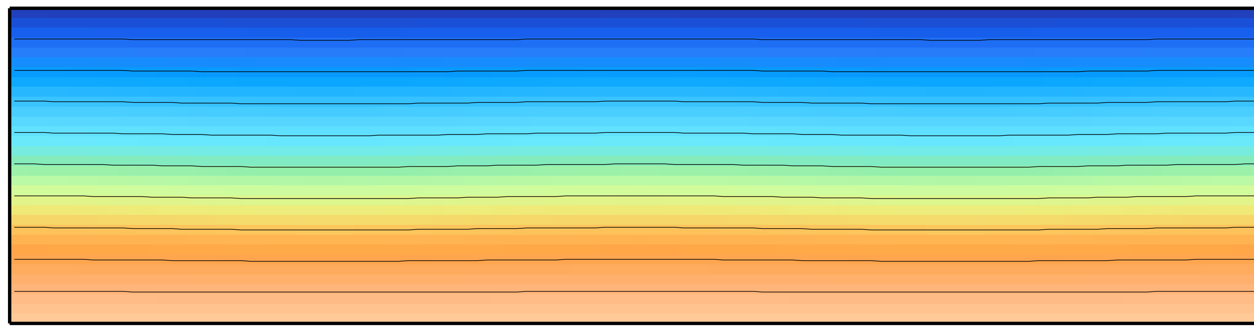
Results 1: Onset of convection of isoviscous fluid

Convective Instability: Grow or Not ?

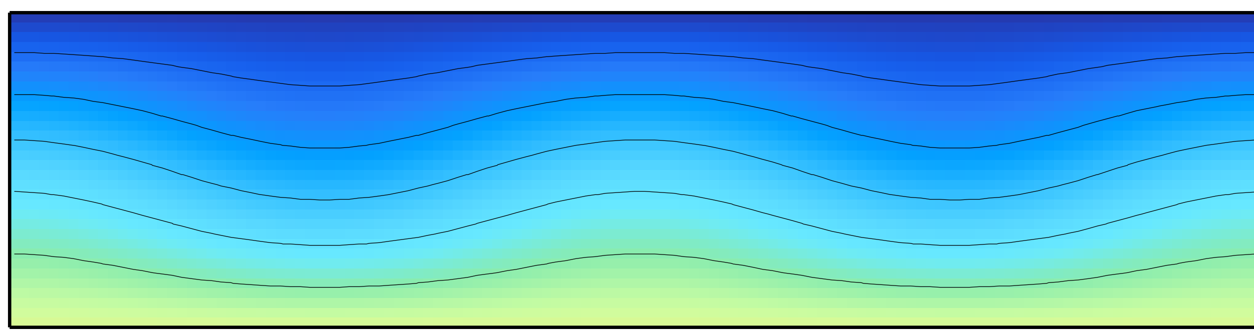
Convective instability grows up for sufficiently Rayleigh number Ra

$$Ra \propto \frac{\boxed{\text{thermal buoyancy}}}{\boxed{\text{viscous resistance}} \times \boxed{\text{thermal diffusion}}}$$

$Ra = 1000$

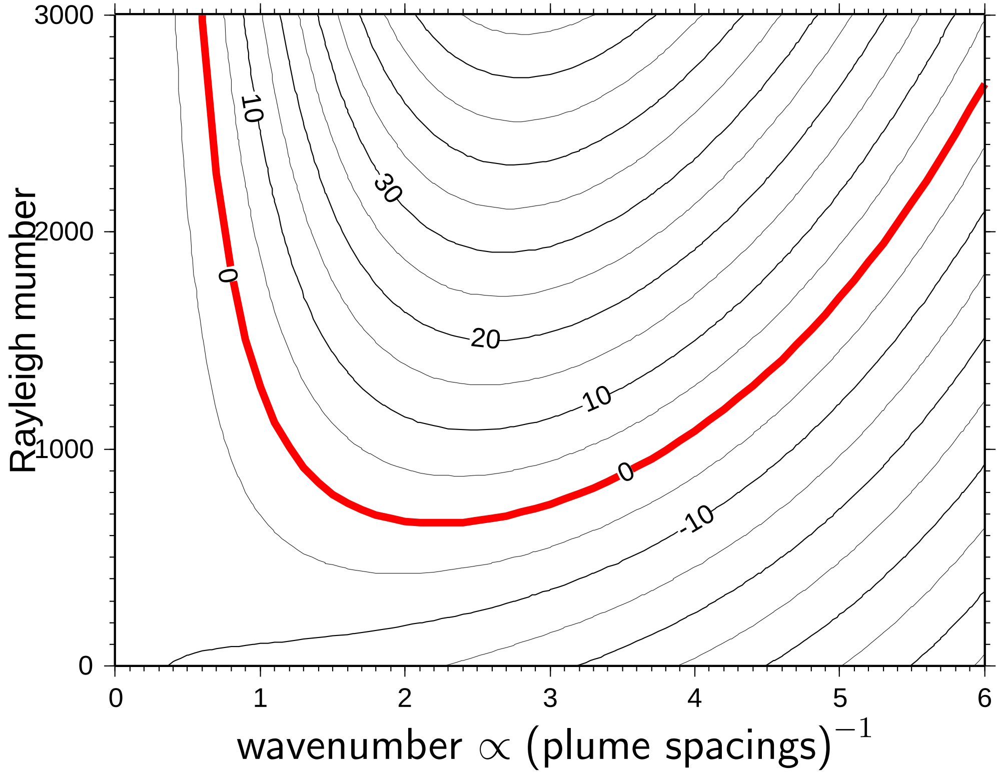


$Ra = 600$



Temporal evolution of perturbation

Growth rate a of convective instability (perturbation)

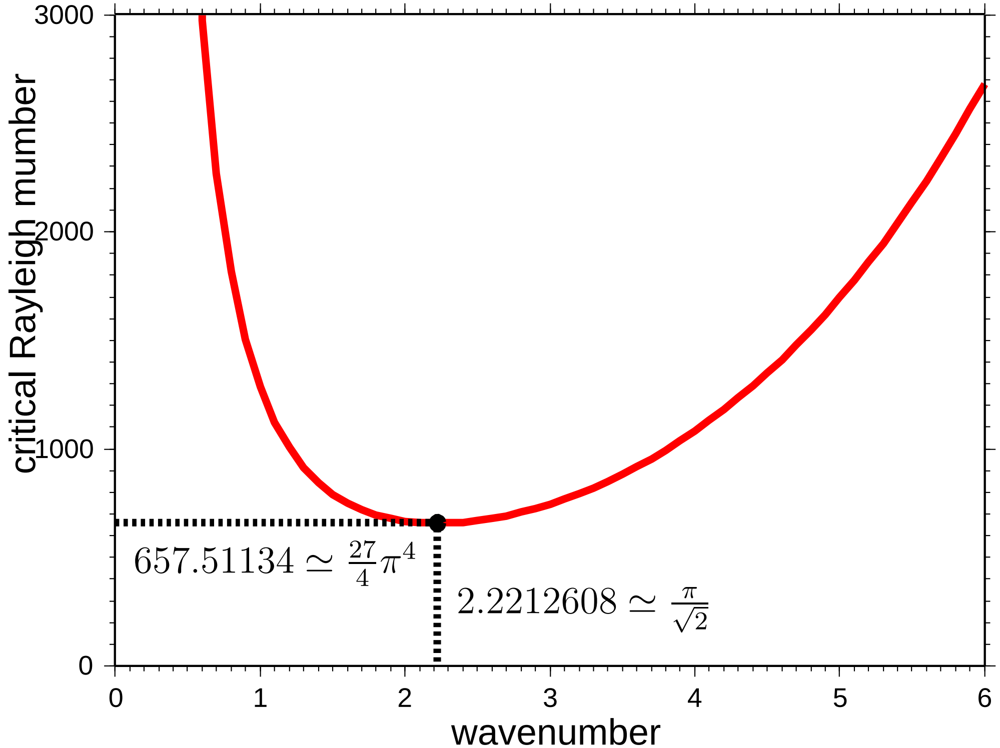


$a > 0$ for growth,
$a < 0$ for decay,
$a = 0$ for neutral.

Growth rate a becomes:

- ❑ larger for larger Rayleigh number Ra
- ❑ largest for intermediate wavenumber K of perturbation

Critical Rayleigh number (Ra_c): Ra which gives $a = 0$.

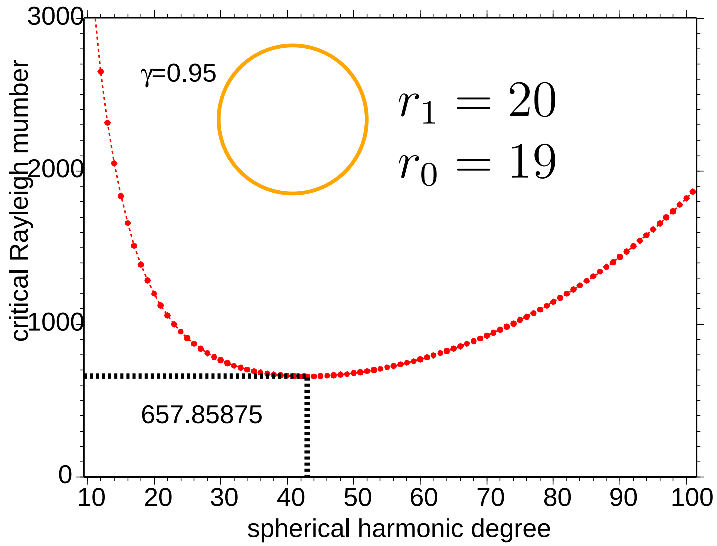


$a > 0$ for growth,
$a < 0$ for decay,
$a = 0$ for neutral.

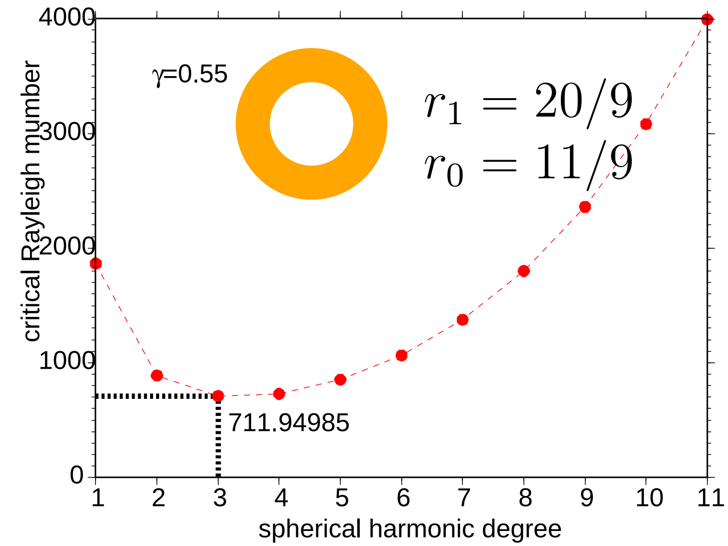
The perturbation with absolute minimum of Ra_c ($\equiv Ra_{c0}$) is most important. (because it is destabilized most easily !)

Critical Rayleigh number in spherical shells

thin shell ($\gamma = 0.95$)



thick shell ($\gamma = 0.55$)



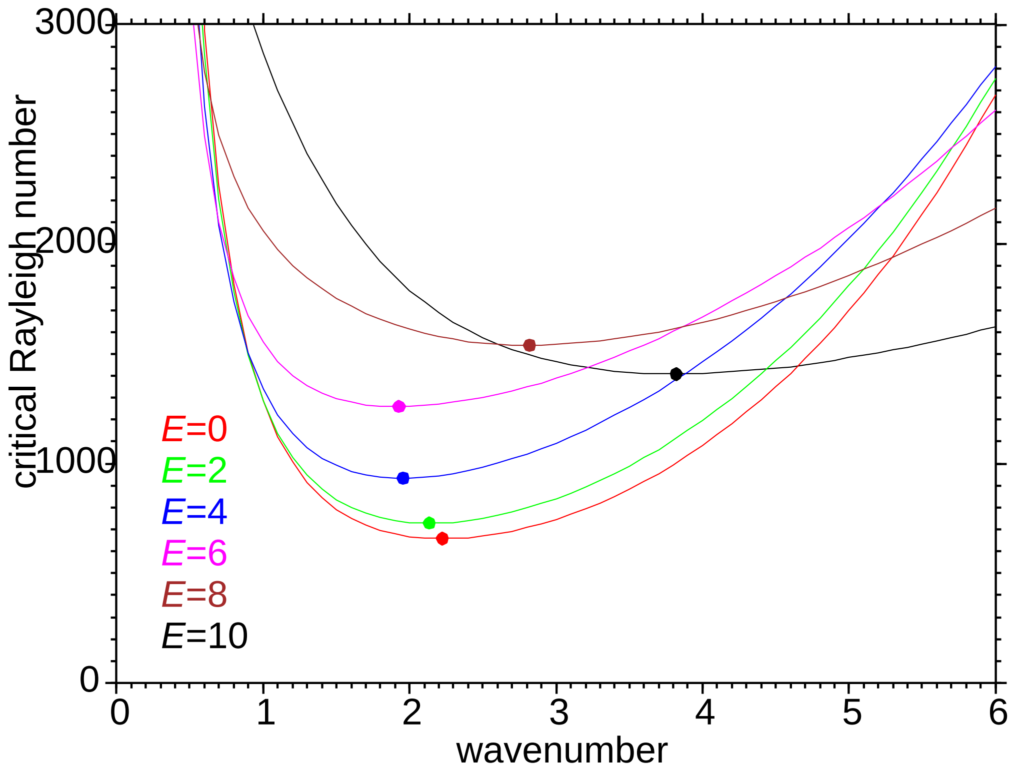
Note: Length scale is normalized so as to take $r_1 - r_0 \equiv 1$.

Growth of perturbation depends on

- ❏ Rayleigh number Ra
- ❏ spherical harmonic degree ℓ of perturbation
- ❏ thickness (or aspect ratio γ) of spherical shell

Results 2: Onset of convection of temperature-dependent viscosity fluid in a planar layer

Variations in critical Rayleigh numbers

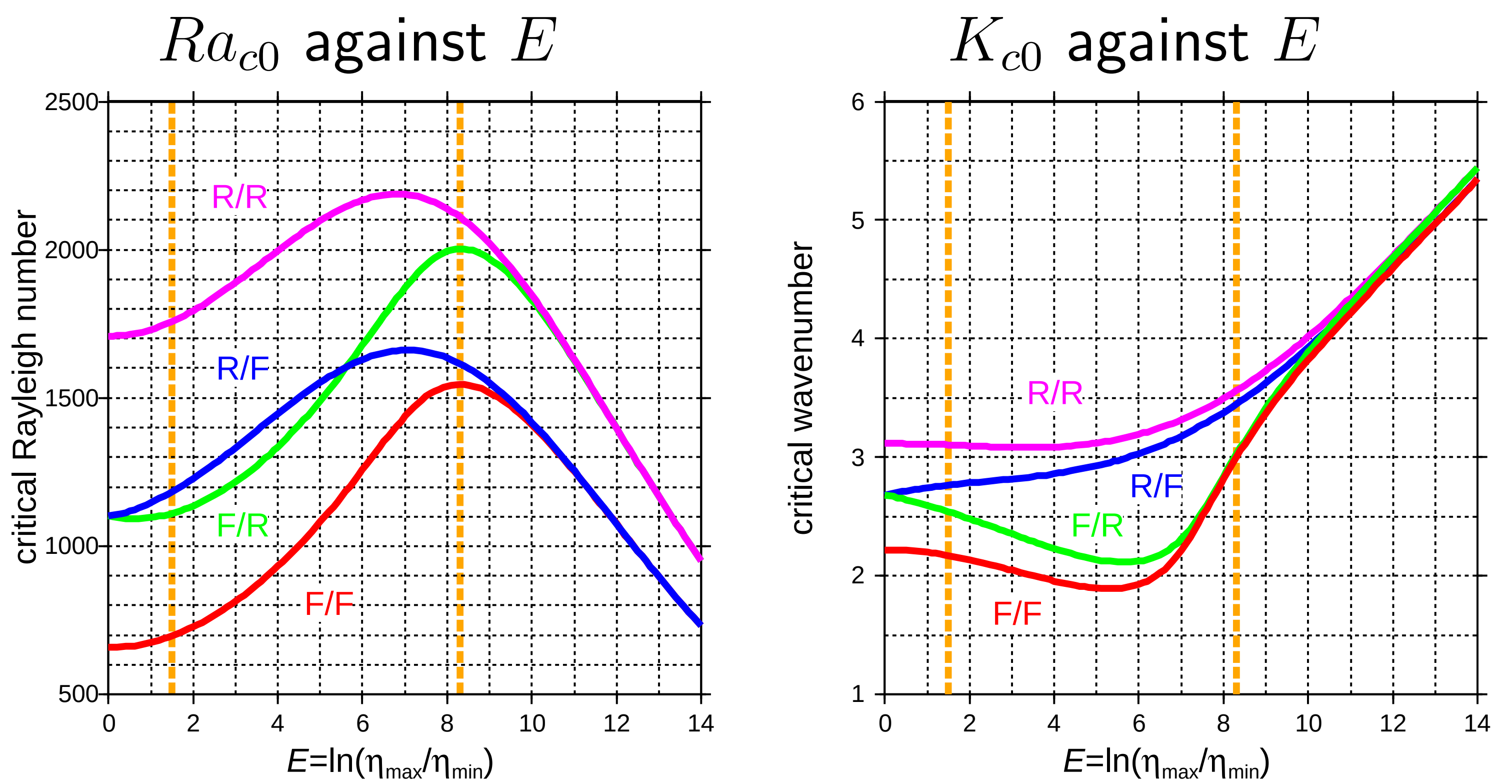


variations in critical Rayleigh number Ra_c and wavenumber K with increasing temperature dependence of viscosity E.

$$E = \ln(\eta_{\max}/\eta_{\min})$$

- Variations of Ra_c on K depend on E,
- Absolute minimum of Ra_c ($\equiv Ra_{c0}$) and corresponding K ($\equiv K_{c0}$) are also dependent on E.

Critical Rayleigh number and wavenumber (1)



Three regimes can be identified depending on E:

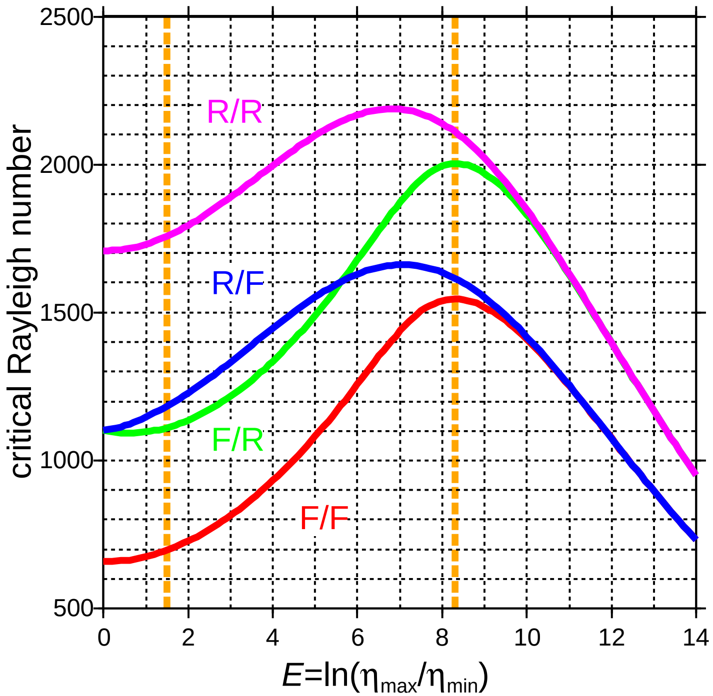
$E \lesssim 1.5$: $Ra_{c0} \rightarrow$ and $K_{c0} \rightarrow$

$1.5 \lesssim E \lesssim 8.3$: $Ra_{c0} \nearrow$ and $K_{c0} \rightarrow$ or $\searrow$ (depending on b.c.)

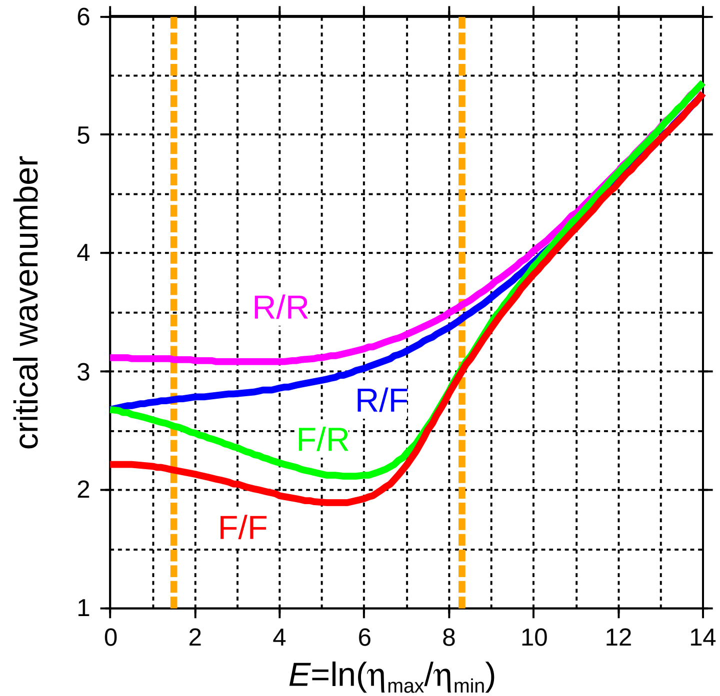
$E \gtrsim 8.3$: $Ra_{c0} \searrow$ and $K_{c0} \nearrow$ Stagnant-Lid mode

Critical Rayleigh number and wavenumber (2)

Ra_{c0} against E



K_{c0} against E

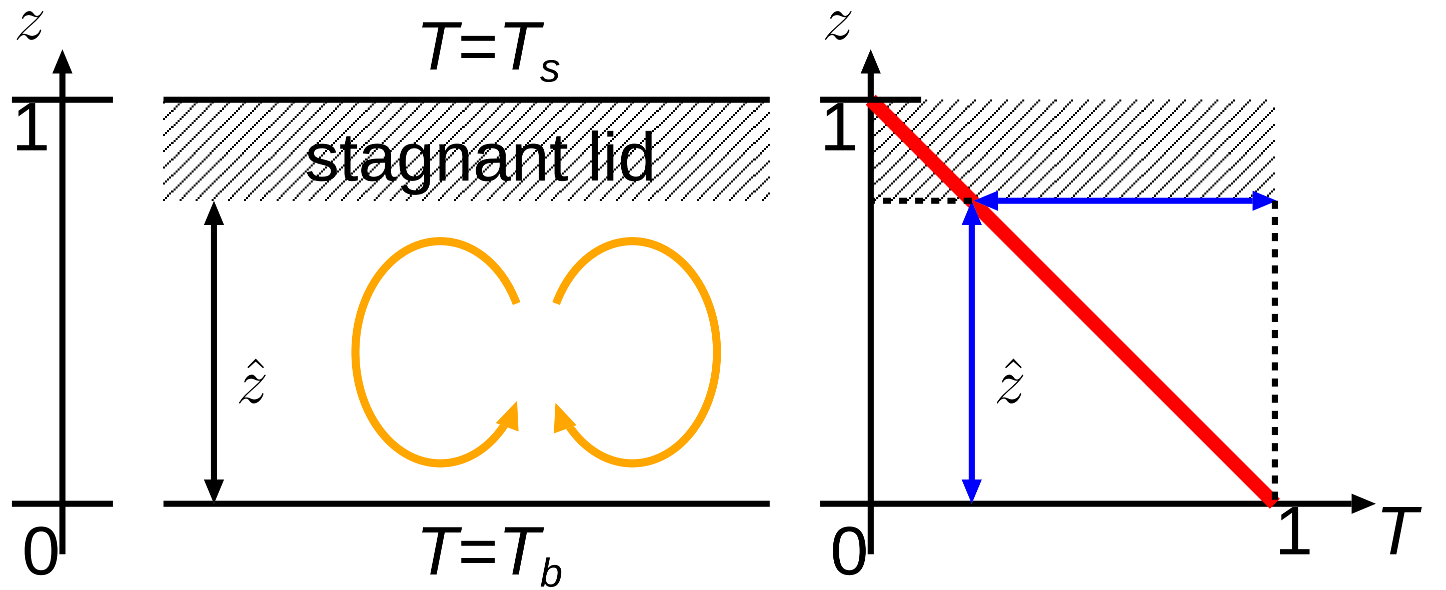


For very strong temperature-dependence (large E),

- ❑ differences in top surface boundary conditions become negligibly small.
  - ➠ almost zero motion even with free-slip top boundaries.

Analytical Criterion for ST-mode (1)

Following the idea by Stengel *et al.* (1982) ....

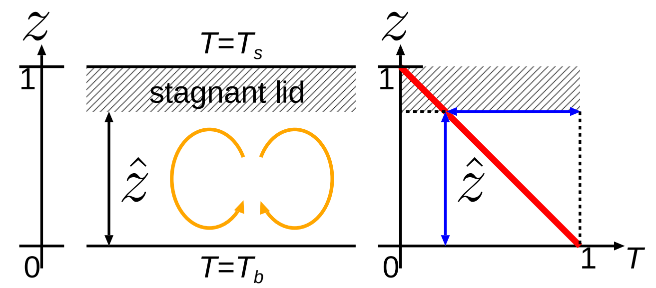


In ST regime, convection begins in a basal sublayer, not in an entire layer.

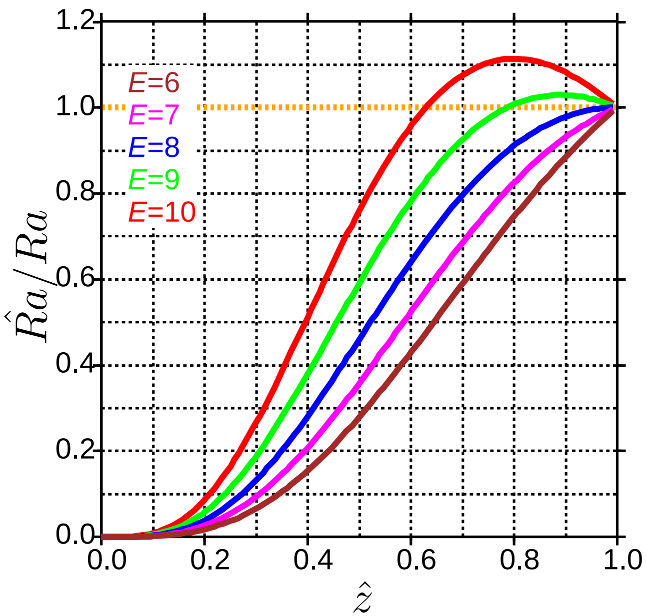
For sufficiently large E, the Rayleigh number $\hat{Ra}$ local to the basal sublayer exceeds Ra for the entire layer.

Analytical Criterion for ST-mode (2)

Consider the ratio of sublayer's $\hat{Ra}$ to Ra, in order to confirm the transition into ST regime at $E \simeq 8$.



$$\frac{\hat{Ra}}{Ra} = \frac{\boxed{\text{thickness}}^3 \times \boxed{T\text{-gap}}}{\boxed{\text{viscosity}}} = \hat{z}^4 \exp\left[E\left(\frac{1}{2} - \frac{\hat{z}}{2}\right)\right]$$

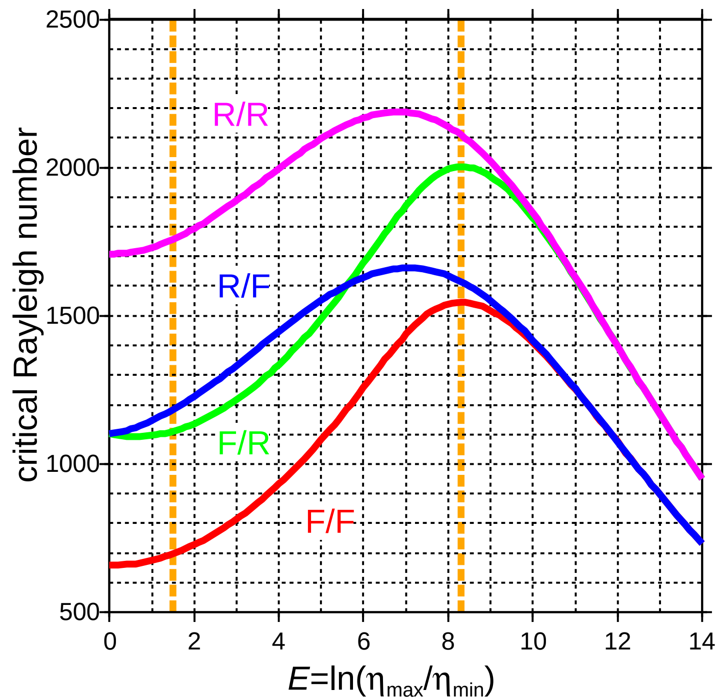


When $E > 8$ there exists a range of $\hat{z}$ (< 1) which allows $\hat{Ra} > Ra$.

Convective instability tends to grow only in a basal sublayer with $\hat{Ra} > Ra$

$\Rightarrow$ "Stagnant-Lid" mode

Analytical Criterion for ST-mode (3)



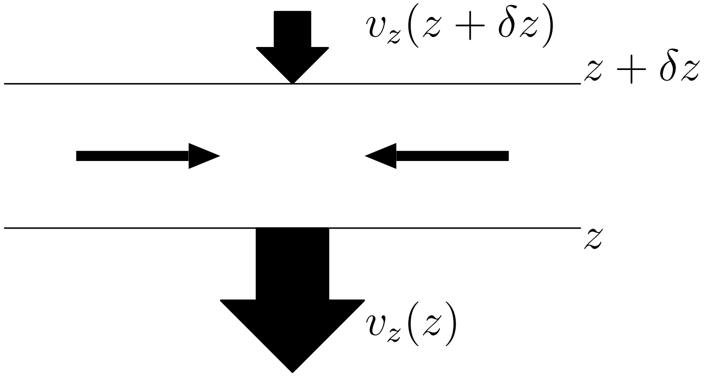
Above analytical consideration also suggests that Ra for the entire layer should be maximum at the transition into ST regime.

The plots for free-slip top surfaces (F/F and F/R) have maxima at around $E = 8.3$.

$\Rightarrow$ Transition into ST regime occurs at $E \simeq 8.3$ for a planar layer.

Empirical Criterion for ST-mode (1)

Considering where descending flows are amplified most intensively ....

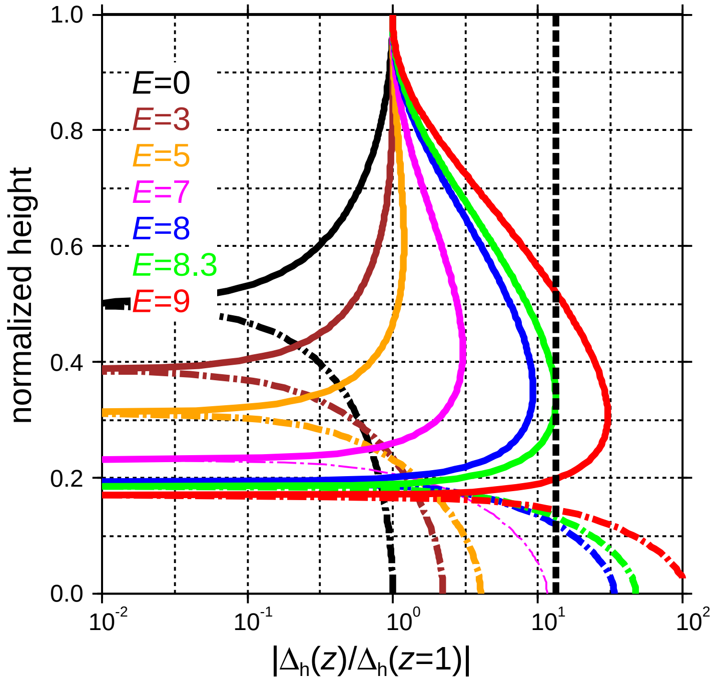


Here we define a quantity Δ_h as a function of height z defined with,

$$v_z(z) - v_z(z+\delta z) \simeq -\frac{\partial v_z}{\partial z}\delta z = \underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)}_{\propto \Delta_h} \delta z$$

The value of $|\Delta_h|$ becomes locally maximum at a height z where a vertical flow is amplified most intensively.

Empirical Criterion for ST-mode (2)



Vertical profiles of $|\Delta_h|$ for various E

As for a local maximum other than at $z = 0$,

- ❑ Where does it occur?
- ❑ How large is it ?

The local maximum of $|\Delta_h|$ occurs at deeper position with increasing E.

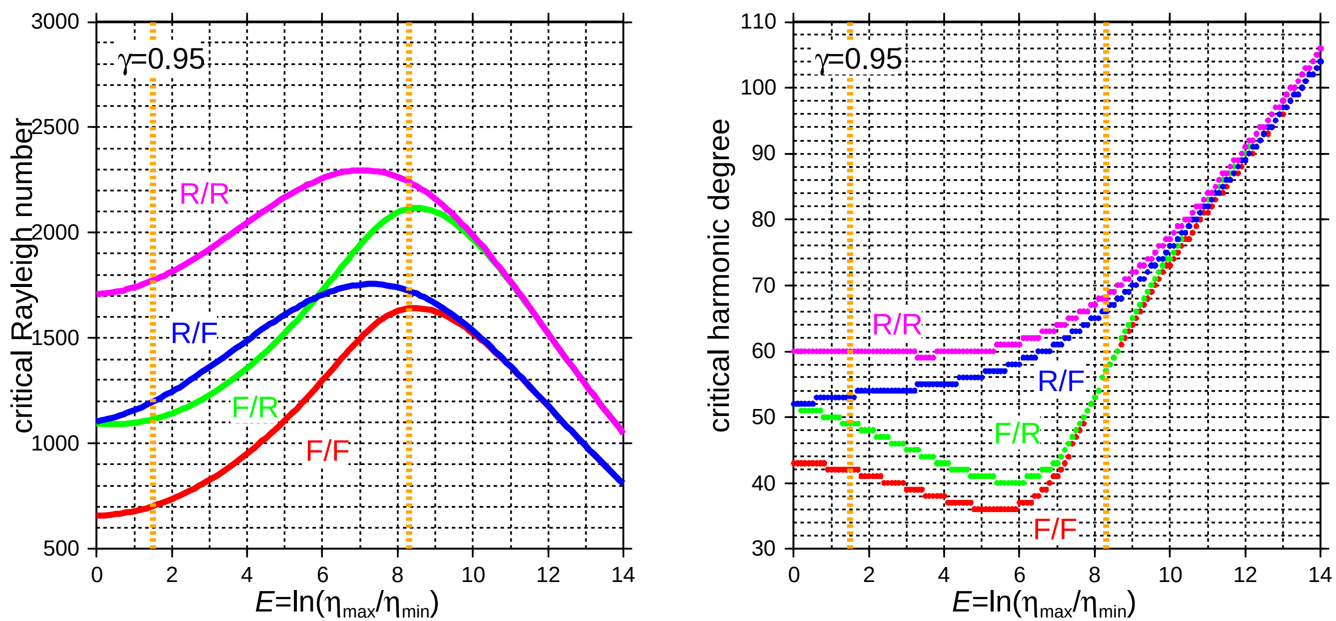
At transition into ST regime ($E = 8.3$), the local maximum becomes more than 10 times larger than $|\Delta_h|$ at $z = 1$.
$\Rightarrow$ to be used as a criterion for a spherical shell convection.

Results 3: Onset of convection of temperature-dependent viscosity fluid in a spherical shell

Critical Rayleigh number and wavenumber (1)

Thin spherical shell ($\gamma = 0.95$)

Ra_{c0} against E $\qquad$ ℓ_{c0} against E



$r_1 = 20$
$r_0 = 19$

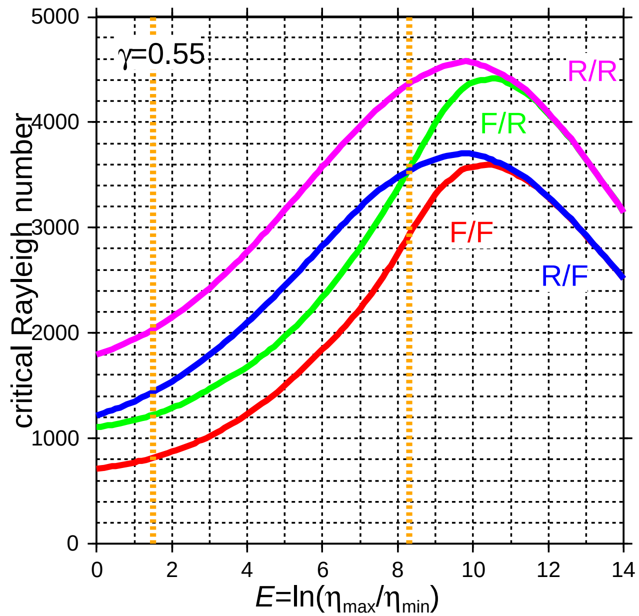
Quite similar to the case in a planar layer:

- Three regimes (small E, moderate E, and large E),
- Maxima in the plots of Ra_{c0} for F/F and F/R at $E \simeq 8.3$,
- smooth changes in Ra_{c0} and ℓ_{c0}.

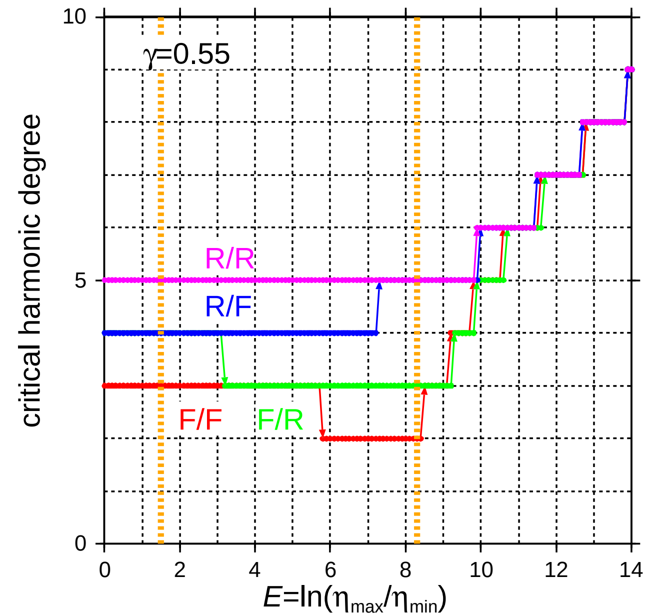
Critical Rayleigh number and wavenumber (2)

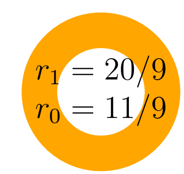
Thick spherical shell ($\gamma = 0.55$)

Ra_{c0} against E



ℓ_{c0} against E

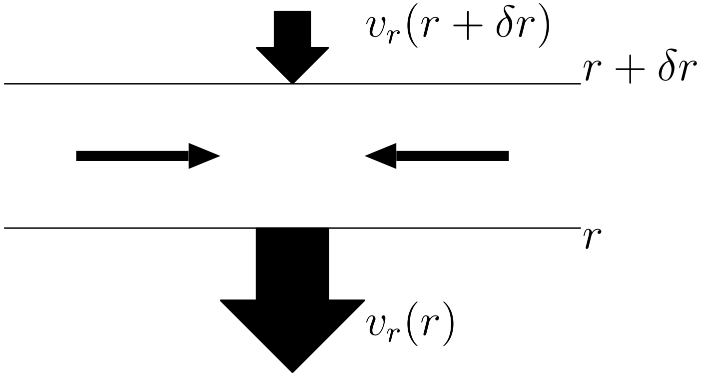




Three regimes (small E, moderate E, and large E) exist, but ...

- Maxima in the plots of Ra_{c0} for F/F and F/R at $E \simeq 10$, (but NOT the transition into ST-mode)
- discontinuous changes in Ra_{c0} and ℓ_{c0}. (mostly due to small horizontal extent)

Empirical Criterion for ST-mode (1)



Suppose again where descending flows are amplified most intensively in a spherical shell.

Here we consider a quantity Δ_h as a function of radius r defined with,

$$r^2 v_r(r) - (r+\delta r)^2 v_r(r+\delta r) \simeq -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) r^2 \delta r$$

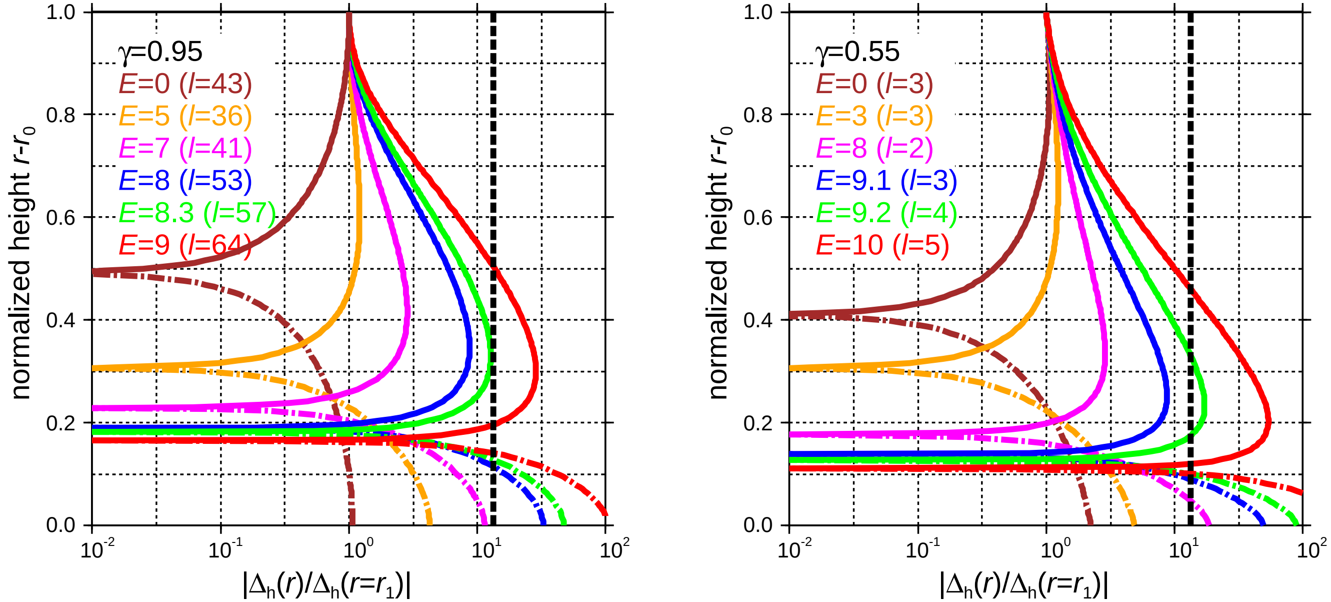
$$= \underbrace{\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \right)}_{\propto \Delta_h} r^2 \delta r$$

Note the definition of Δ_h different from previous one.

Empirical Criterion for ST-mode (2)

(a) thin shell ($\gamma = 0.95$)

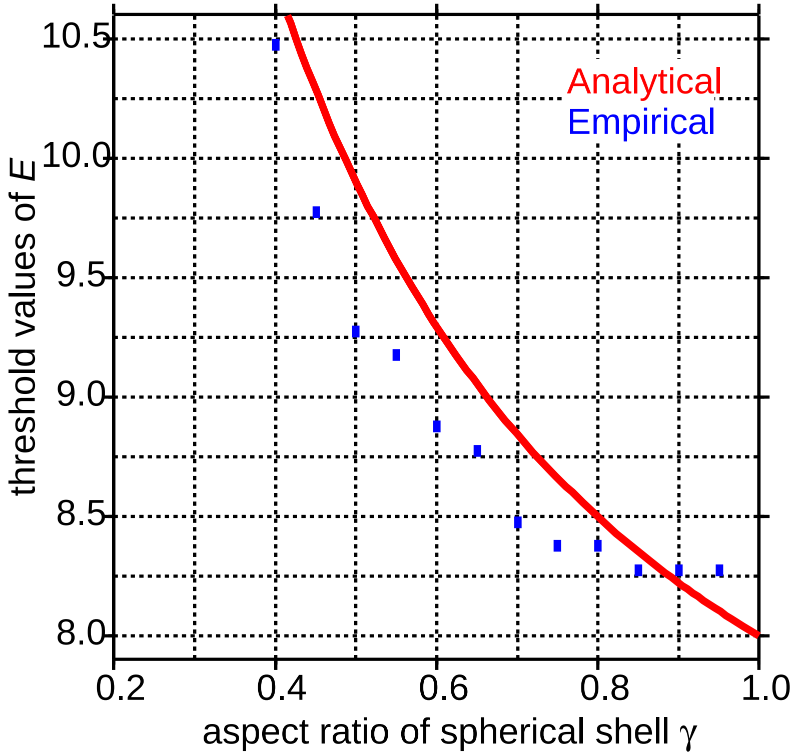
(b) thick shell ($\gamma = 0.55$)



Based on similar criterion with a planar layer, the transition into ST-mode occurs

- at $E \simeq 8.3$ for $\gamma = 0.95$ (as in a planar layer)
- at $E \simeq 9.2$ for $\gamma = 0.55$

Empirical Criterion for ST-mode (3)



The variations against the aspect ratio of the spherical shell γ of the values of E_c for the transition into the ST regimes.

blue : empirical estimate

(from Δ_h)

red : analytical estimate

(although not shown in detail)

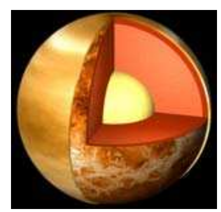
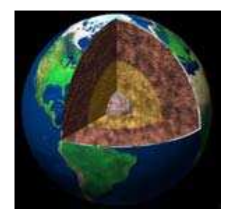
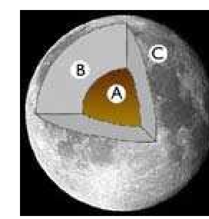
In general, smaller γ (thicker shell) needs larger E for the transition into ST regimes.

❑ because of larger temperature contrast between ascending flows and surroundings ?

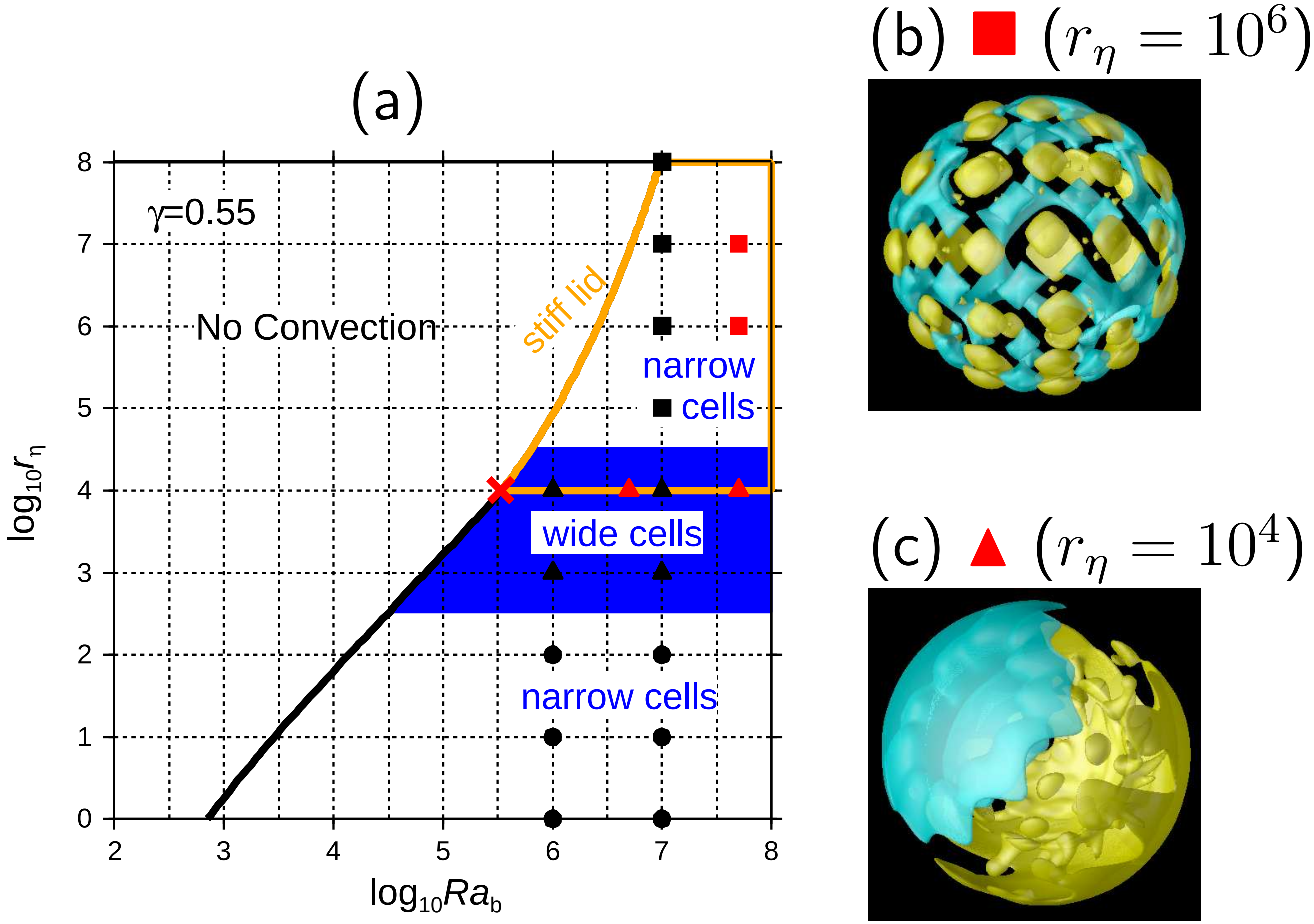
Discussion and Concluding Remarks

Application to mantle convection in other bodies?

Planetary Data

	Mercury	Venus	Earth	Mars	Moon
Radius	0.38	0.95	1	0.54	0.27
Mass	0.055	0.815	1	0.107	0.012
Density [kg/m^3]	5430.	5250.	5515.	3940.	3340.
MoI	0.34	?	0.3355	0.3662	0.3905
Rc/Rp	0.8	0.55?	0.546	0.5	0.25
					
E_c	8.4	9.2	9.2	9.3	> 10.5

Regime diagram for Earth's mantle: Revisited (1)

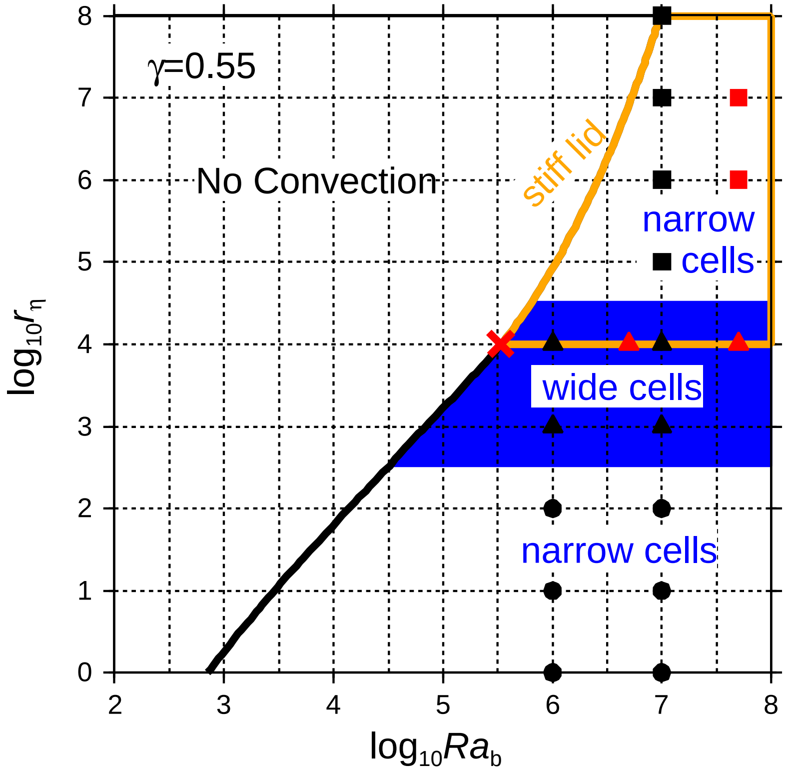


The curve of critical Rayleigh number Ra_b and viscosity contrast $r_\eta \equiv \exp(E)$ has a bend near × $(r_\eta = 10^4 \simeq e^{9.2})$.

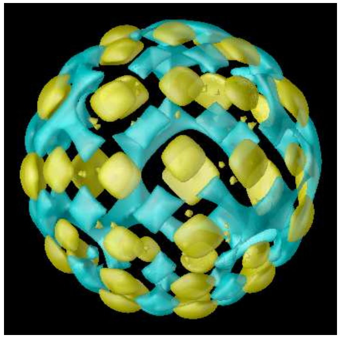
$\Rightarrow$ demonstrates significance of present estimate

Regime diagram for Earth's mantle: Revisited (2)

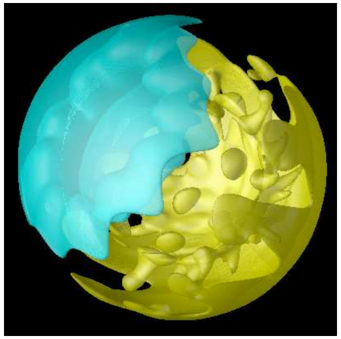
(a)



(b) ■ ($r_\eta = 10^6$)



(c) ▲ ($r_\eta = 10^4$)



For the value of $r_\eta = 10^4$ at the transition into ST regime, convection has horizontally-elongated cells.

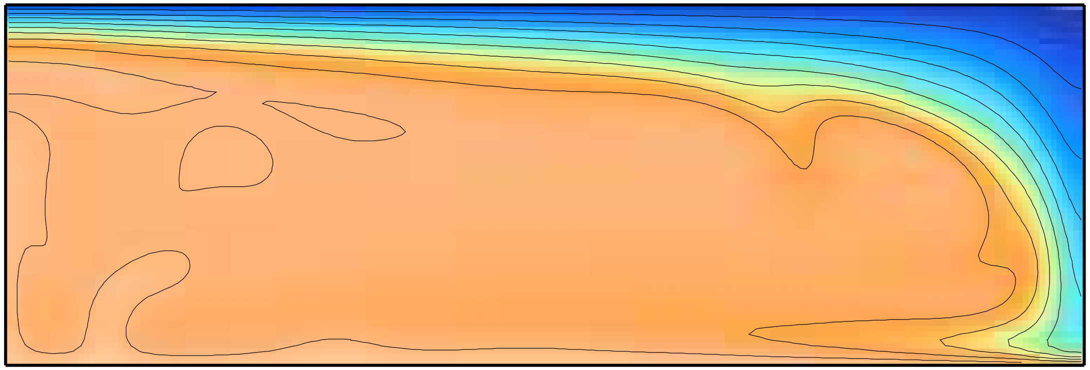
wide convection cells beneath cold stiff lids at $r_\eta \simeq 10^4$?

ST-mode of convection with elongated cells

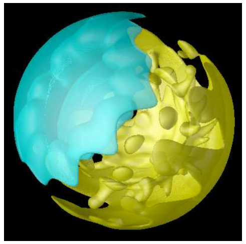
"Elongated-ST" mode

(figure and movie modified from Kameyama and Ogawa, 2000)

$Ra_{\mathrm{b}} = 6 \times 10^6$, $r_\eta = 10^4$, width/height=3



$\simeq$



- ❑ horizontally-elongated convection cell
- ❑ minor descending plumes from base of cold lid
  - ➠ cold lid is stiff enough to prevent minor instabilities from penetrating upward.

Future directions

How come the convective flows with wide cells beneath cold stiff lids in mantles of terrestrial planets ?

In particular,

what mechanisms cause convection cells of large horizontal length scales ?

Hopefully, graduate students will address:

❑ the effects of material properties (other than viscosity) ?

➨ thermal expansivity, thermal conductivity, ... ?

❑ the effects of chemical heterogeneity (and surface tectonics) ?

➨ If ascending plumes are “anchored” by chemical “piles” in the lowermost mantle ?

Linear Stability Analysis

Linearization (1)

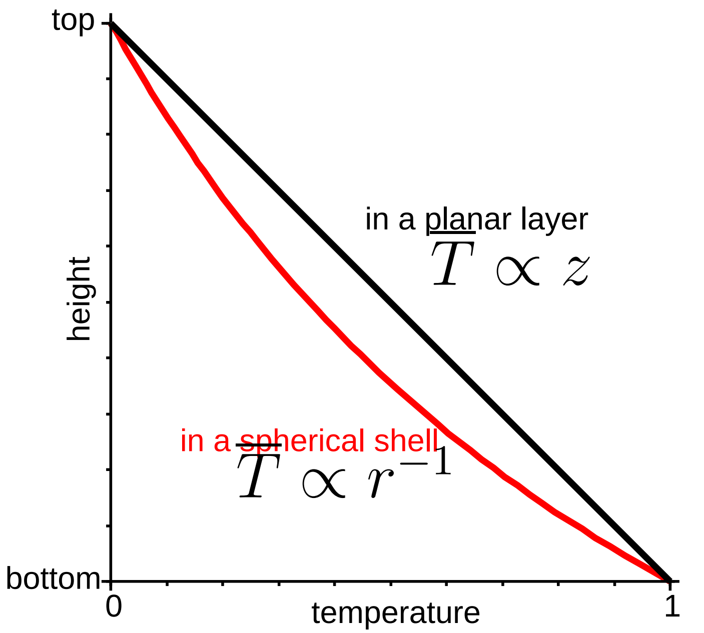
Split all quantities into the sum of

- reference state (denoted by overbars)
- infinitesimal perturbation (denoted by primes)

$$T = \overline{T} + T', \quad \boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}', \quad p = \overline{p} + p'$$

Choice of reference state

- $\boldsymbol{v} = \boldsymbol{0}$ (motionless)
- $0 = \nabla^2 \overline{T}$ (1-D steady heat conduction)
  $0 = \frac{d^2\overline{T}}{dz^2}$ for a planar layer,
  $0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\overline{T}}{dr}\right)$ for a spherical shell,



Linearization (2)

Derive linearized equation for infinitesimal perturbations
Dropping the second-order terms yields

- Equation of heat transport

$$\frac{\partial T'}{\partial t} + \boldsymbol{v'} \cdot \nabla \overline{T} = \nabla^2 T'$$

- Equation of continuity (incompressible fluid)

$$\nabla \cdot \boldsymbol{v'} = 0$$

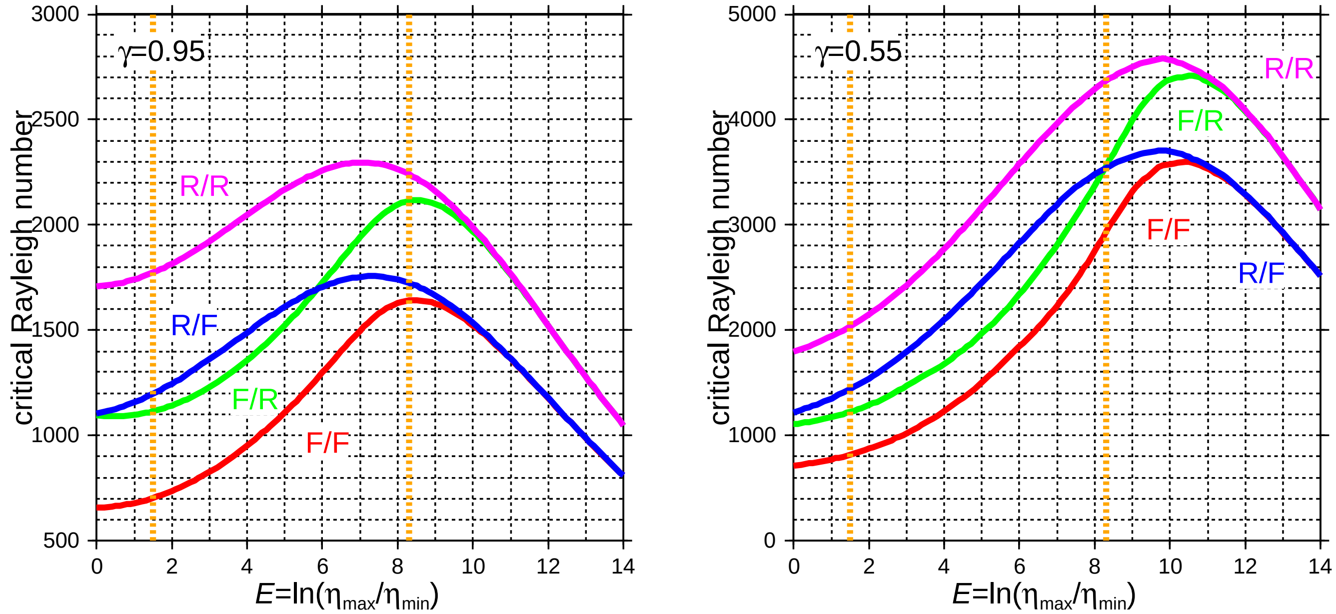
- Equations of motion (force balance)

$$0 = -\nabla p' + \nabla \cdot \left[\overline{\eta} \left(\nabla \otimes \boldsymbol{v'} + \boldsymbol{v'} \otimes \nabla\right)\right] + RaT'\boldsymbol{e}_g$$

Analytical Estimate for Transition into ST regime in a spherical shell

Analytical Criterion for ST-mode (1)

thin shell ($\gamma = 0.95$) thick shell ($\gamma = 0.55$)



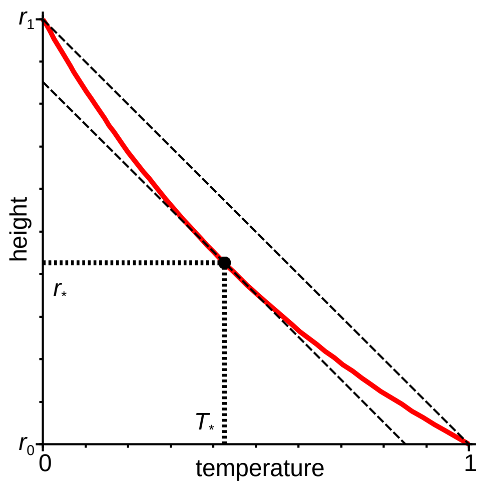
In contrast to a planar case, the maxima of Ra_{c0} against E do not necessarily indicate the transition into ST regimes.

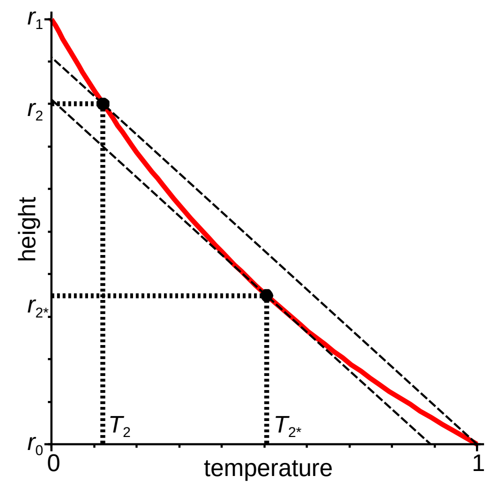
Analytical Criterion for ST-mode (2)

By using appropriately defined Rayleigh number Ra_*, we will explore the analytical estimate for the transition.

Key assumption:

(a) r_* and T_* (in entire layer) (b) r_{2*} and T_{2*} (in lower sublayer)

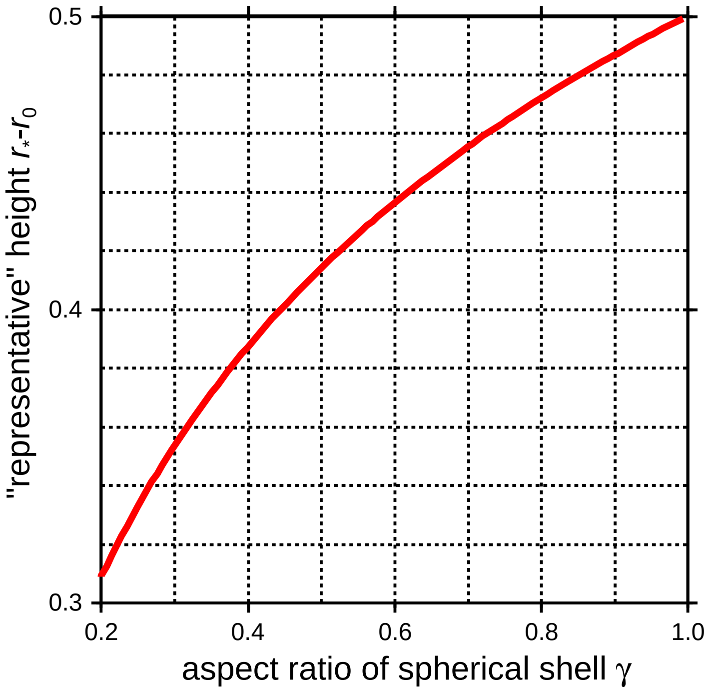




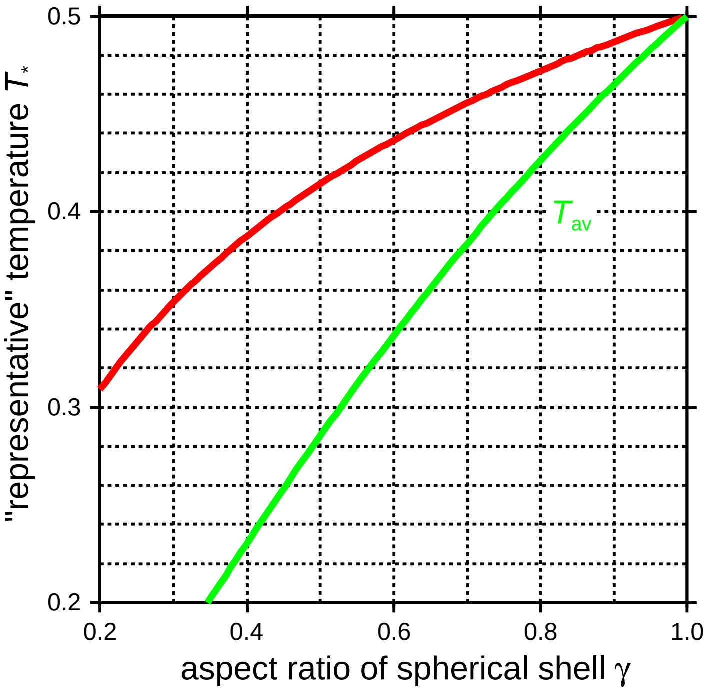
The temporal evolution of perturbation is assumed to be sensitive to the viscosity of $T = T_*$ at $r = r_*$ where the radial temperature gradient is equal to an average temperature gradient across the entire layer.

Analytical Criterion for ST-mode (3)

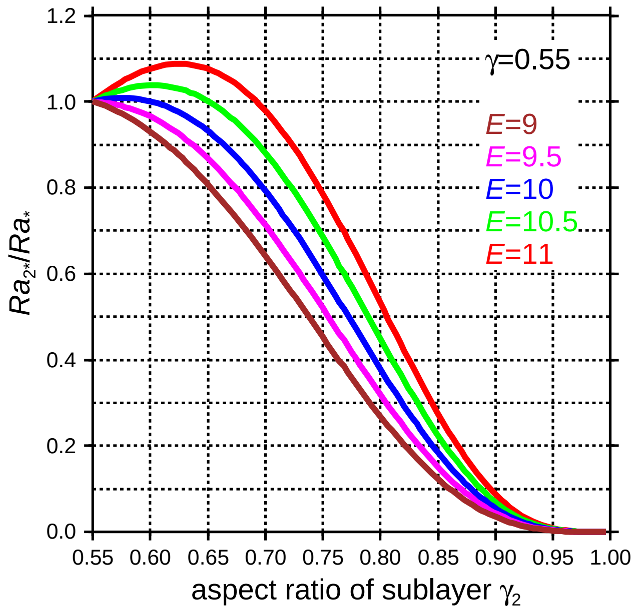
(a) $r_* - r_0$ against γ



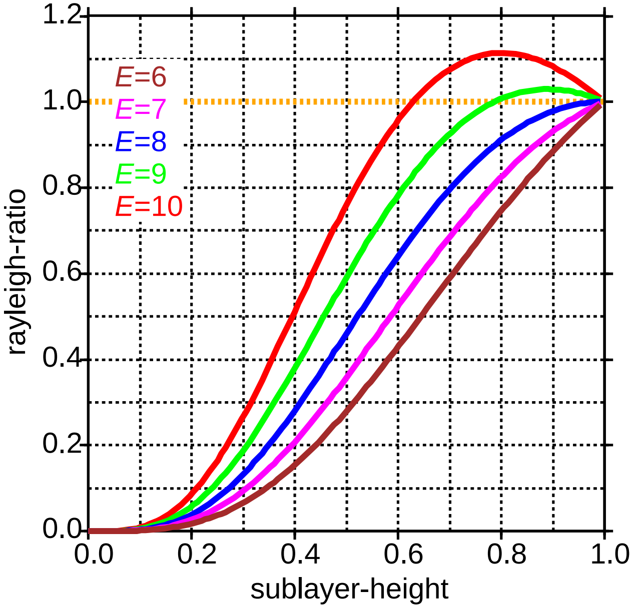
(b) T_* against γ



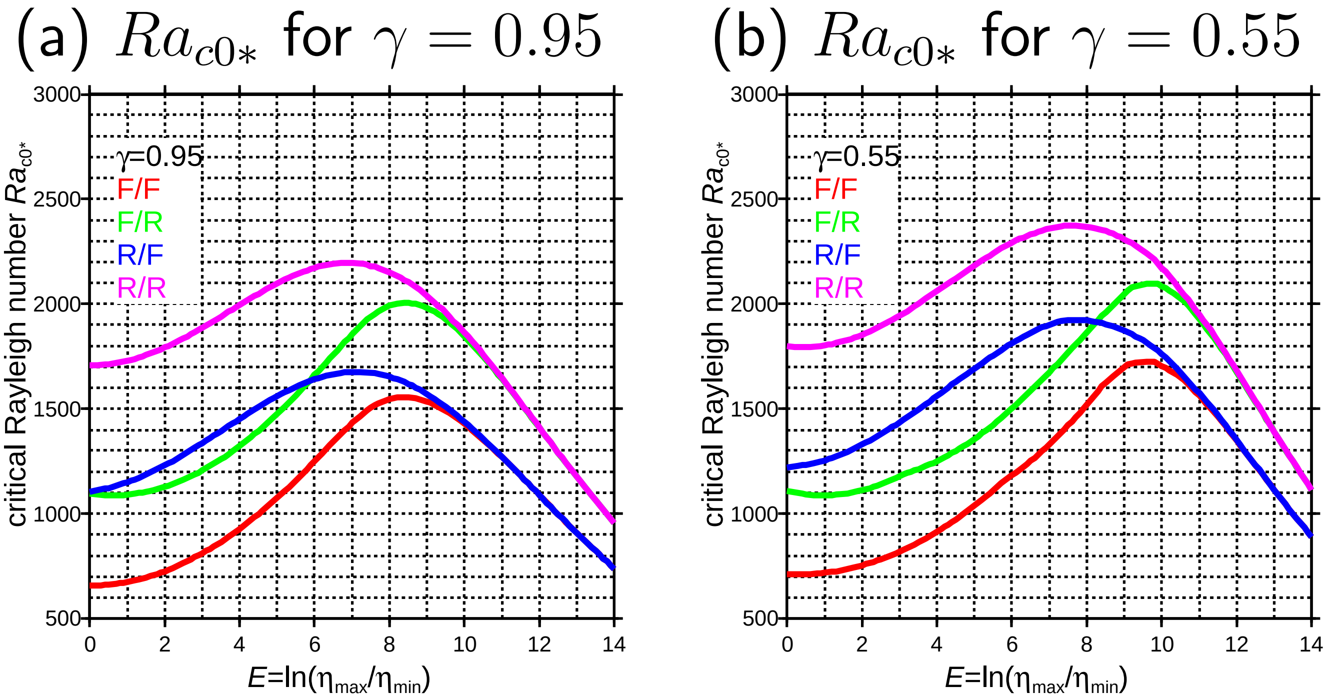
(a) $\gamma = 0.55$



(b) planar layer



Analytical Criterion for ST-mode (5)



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