A One-Fluid MHD Model with Electron Inertia

Keiji Kimura[1)] and Philip J. Morrison[2)]

[1)]Kyoto University, [2)]The University of Texas at Austin

Geophysical Fluid Dynamics Program 2011

(20th June 2011—26th August 2011)

Woods Hole Oceanographic Institution

Geophysical Fluid Dynamics Program 2011 at Woods Hole Oceanographic Institution

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Menu

- Research Project in GFD Program 2011
- Lectures in GFD Program 2011
- Life at GFD

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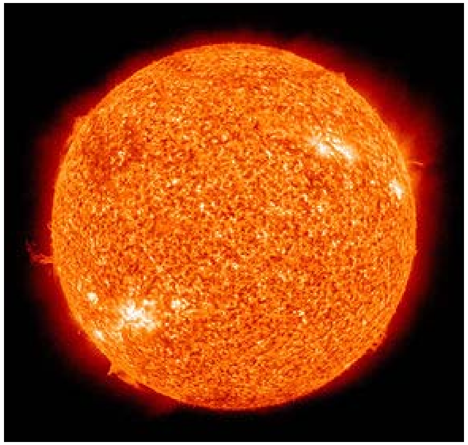
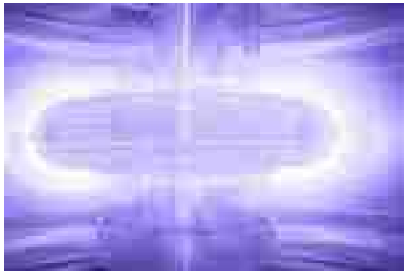
- Introduction
- Linear Wave Modes
- Energy Conservation
- Equilibrium States
- Conclusion
- Future Works

Introduction

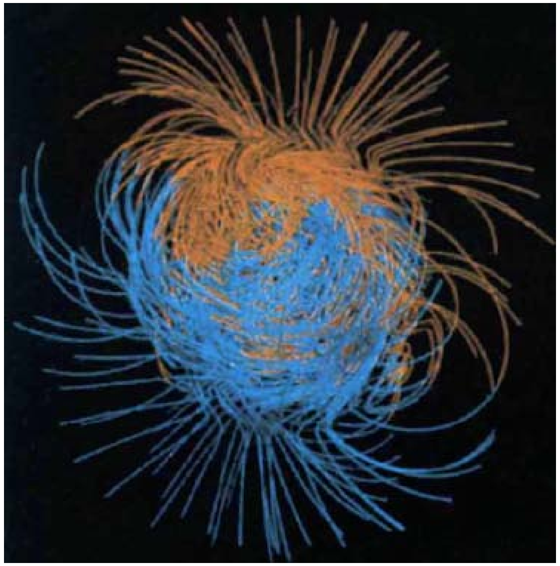
- Magnetohydrodynamic (MHD) approximation is well-used.



(Wikipedia Sun)



(HP of Prof. Z. Yoshida)



(Glatzmeier and Roberts, 1995)

- What are the limitations?

Two Fluid MHD Model

- Kinetic theory

→ Two-fluid model (taking moments)

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \boldsymbol{V}_i) = 0,$$

$$m_i n_i \left(\frac{\partial \boldsymbol{V}_i}{\partial t} + (\boldsymbol{V}_i \cdot \nabla) \boldsymbol{V}_i \right) + \nabla \cdot \overline{\boldsymbol{p}_i} - q_i n_i (\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}) = -\boldsymbol{F},$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \boldsymbol{V}_e) = 0,$$

$$m_e n_e \left(\frac{\partial \boldsymbol{V}_e}{\partial t} + (\boldsymbol{V}_e \cdot \nabla) \boldsymbol{V}_e \right) + \nabla \cdot \overline{\boldsymbol{p}_e} - q_e n_e (\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}) = \boldsymbol{F},$$

- Quasi-neutrality

$$\boxed{\begin{aligned} q_i &= -q_e \equiv e, \\ n_i &= n_e \equiv n \end{aligned}}$$

$$\rho \equiv (m_e + m_i) n$$

$$\boldsymbol{V} \equiv \frac{m_e \boldsymbol{V}_e + m_i \boldsymbol{V}_i}{m_e + m_i}, \qquad \boldsymbol{j} \equiv en(\boldsymbol{V}_i - \boldsymbol{V}_e)$$

One Fluid MHD Model

- Kinetic theory

→ Two-fluid model (taking moments)

→ One-fluid model (quasi-neutrality) (Lüst, 1959)

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}),$$

: Continuity eqn.

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla\cdot\overline{\boldsymbol{p}} + \boldsymbol{j}\times\boldsymbol{B} \boxed{-\frac{m_e}{e}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en}},$$

: Momentum eqn.

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \boxed{\frac{1}{\sigma}\boldsymbol{j}} + \boxed{\frac{1}{en}(\boldsymbol{j}\times\boldsymbol{B} - \nabla\cdot\overline{\boldsymbol{p}_e})}$$

Hall term

Collision (resistivity)

$$\boxed{\begin{aligned} &+\frac{m_e}{e^2 n}\left[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla\cdot(\boldsymbol{V}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{V})\right] \\ &-\frac{m_e}{e^2 n}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en}, \end{aligned}}$$

: Generalized
Ohm's law

Electron inertia

Ohm's Law in Previous Studies

Vasyliunas (1975)

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = \eta\mathbf{J} + \frac{m_e}{ne^2}\left[\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{JV} + \mathbf{VJ})\right]$$

$$- \frac{1}{ne}\nabla\cdot\mathbf{P}^{(e)} + \frac{1}{nec}\mathbf{J}\times\mathbf{B}$$

Fitzpatrick (2001)

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} \simeq \frac{\mathbf{F}_U}{ne} + \frac{\mathbf{j}\times\mathbf{B}}{ne} + \frac{m_e}{ne^2}\frac{d\mathbf{j}}{dt}$$

$$+ \frac{m_e}{ne^2}(\mathbf{j}\cdot\nabla)\mathbf{V} - \frac{m_e}{n^2e^3}(\mathbf{j}\cdot\nabla)\mathbf{j}$$

Watson

$$\mathbf{R} = \frac{\mathbf{J}}{\sigma} + \frac{1}{nec}(\mathbf{J}\times\mathbf{B} - \nabla\cdot\mathbf{p}_e) + \frac{m_e}{ne^2}\left(\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{vJ} + \mathbf{Jv})\right)$$

Bhattacharjee et al. (1999)

$$\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = \eta\mathbf{J} + \frac{4\pi}{\omega_{pe}^2}\frac{D\mathbf{J}}{Dt} - \frac{\nabla p}{ne} + \frac{\mathbf{J}\times\mathbf{B}}{nec}$$

Shay et al. (2001)

$$\frac{4\pi}{\omega_{pe}^2}\frac{d\mathbf{J}}{dt} = \mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} - \frac{1}{nec}\mathbf{J}\times\mathbf{B} + \frac{1}{ne}\nabla\cdot\vec{\vec{\mathbf{P}}}_e - \eta\mathbf{J},$$

There is no $\frac{m_e}{e}(j\cdot\nabla)\frac{j}{en}$ term
in the momentum equation!

Classification of MHD Models

$$Re_m \equiv \frac{\text{Nonlinear}}{\text{Collision}} = \sigma\mu_0 UL, \qquad C_H \equiv \frac{\text{Hall}}{\text{Collision}} = \frac{\sigma B}{en},$$

$$C_I \equiv \frac{\text{Electron inertia}}{\text{Collision}} = \frac{\sigma m_e}{e^2 n\tau},$$

- (Ideal) MHD

$$Re_m \gg 1, \quad C_H, C_I \lesssim 1, \qquad \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = 0,$$

- Hall-MHD

$$Re_m, \ C_H \gg 1, \quad \frac{C_I}{C_H} \ll 1 \qquad \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{1}{en}\boldsymbol{j} \times \boldsymbol{B},$$

- Inertial-MHD (IMHD)

$$Re_m, \ C_I \gg 1, \quad \frac{C_I}{C_H} \gg 1$$

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{m_e}{e^2 n}\left[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{V})\right] - \frac{m_e}{e^2 n}(\boldsymbol{j} \cdot \nabla)\frac{\boldsymbol{j}}{en},$$

Inertial MHD (IMHD) Model

$$\frac{C_I}{C_H} = \frac{m_e}{eB}\frac{1}{\tau} \equiv \frac{1}{\Omega_{Ge}\tau} \gg 1$$: The characteristic timescale << the gyroperiod of electron

(Magnetic reconnection region?)

- Governing eqns.

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}),$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - \frac{m_e}{e}(\boldsymbol{j} \cdot \nabla)\frac{\boldsymbol{j}}{en},$$

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{m_e}{e^2 n}\left[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{V})\right] - \frac{m_e}{e^2 n}(\boldsymbol{j} \cdot \nabla)\frac{\boldsymbol{j}}{en},$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}.$$

Pre-Maxwell equations

Our Purposes

- Comparing the linear wave modes between MHD and IMHD.

- Classifying some IMHD models in terms of the energy conservation.

- Considering the effect of electron inertia using IMHD, especially focusing on some equilibrium states.

- Introduction
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Linear Wave Modes in MHD

- Basic state

$\boldsymbol{B} = \boldsymbol{B}_0, \rho = \rho_0, p = p_0,$: uniform

$\boldsymbol{V} = 0, \boldsymbol{E} = 0$: no flow

- Linearized compressible MHD

$$0 = \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \tilde{\boldsymbol{V}},$$

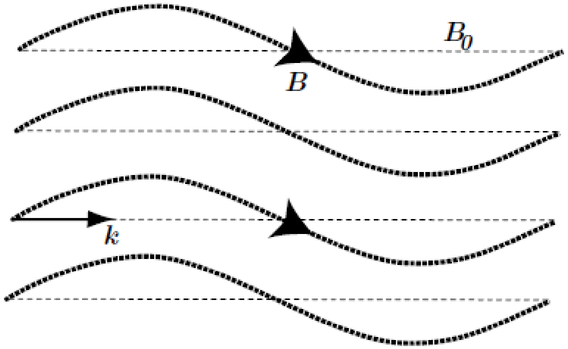
$$\rho_0 \frac{\partial \tilde{\boldsymbol{V}}}{\partial t} = -\nabla \tilde{p} + (\nabla \times \tilde{\boldsymbol{B}}) \times \boldsymbol{B}_0,$$

$$0 = \frac{\partial \tilde{p}}{\partial t} - \gamma \frac{p_0}{\rho_0} \frac{\partial \tilde{\rho}}{\partial t},$$

$$0 = \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{V}} \times \boldsymbol{B}_0,$$

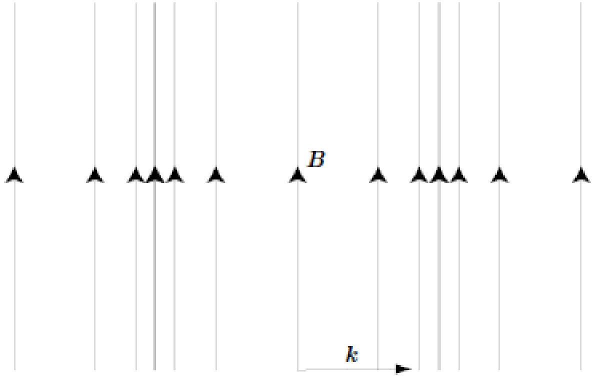
$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = -\nabla \times \tilde{\boldsymbol{E}},$$

(Shear) Alfven wave



$\tilde{B} \perp B_0$

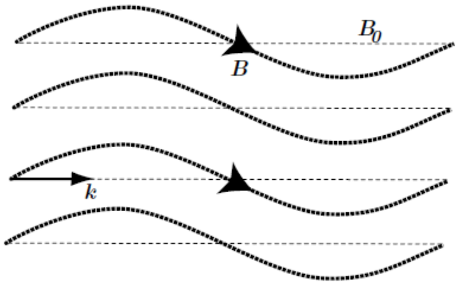
Fast/Slow magnetosonic wave



$\tilde{B} \parallel B_0$

(Fitzpatrick)

Alfven Wave (Morrison and Tassi, 2009)



- MHD

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = -\nabla \times \tilde{\boldsymbol{E}} = \nabla \times (\tilde{\boldsymbol{V}} \times \boldsymbol{B}_0)$$

$$\boxed{\frac{\omega}{k} = V_A \cos\theta}$$

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}}$$: Alfven speed

$$d_e^2 = \frac{m_e}{\mu_0 e^2 n_0} = \frac{c^2}{\omega_{pe}^2}$$

- Inertial MHD

$$\tilde{\boldsymbol{E}} = -\tilde{\boldsymbol{V}} \times \boldsymbol{B}_0 \boxed{+ \mu_0 d_e^2 \frac{\partial \tilde{\boldsymbol{j}}}{\partial t}},$$

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = -\nabla \times \tilde{\boldsymbol{E}},$$

⇨

$$\frac{\partial}{\partial t}(1 - \boxed{d_e^2 \nabla^2}) \tilde{\boldsymbol{B}} = \nabla \times (\tilde{\boldsymbol{V}} \times \boldsymbol{B}_0)$$

$$\boxed{\frac{\omega}{k} = \frac{V_A \cos\theta}{\sqrt{1 + d_e^2 k^2}}}$$

Higher wavenumber waves propagate more slowly!

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Energy Conservation in IMHD

- Governing equations

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}),$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} - \boxed{\epsilon\frac{m_e}{e}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en},}$$

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \boxed{\epsilon\frac{m_e}{e^2 n}\left[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla\cdot(\boldsymbol{V}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{V})\right]} - \boxed{\delta\frac{m_e}{e^2 n}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en},}$$

$$0 = \frac{\partial S}{\partial t} + (\boldsymbol{V}\cdot\nabla)S,$$

$$\nabla\cdot\boldsymbol{B} = 0,$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}.$$

- Energy conservation

$$\begin{aligned}0 = &\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + \rho U + \boxed{\epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{j}|^2}{2}} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\\ &+ \nabla\cdot\left[\left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + p + \rho U + \boxed{\epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{j}|^2}{2}}\right)\boldsymbol{V}\right.\\ &\left. + \epsilon\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\boldsymbol{j})\boldsymbol{j} - \boxed{\delta\frac{m_e}{2e^3 n^2}|\boldsymbol{j}|^2\boldsymbol{j}} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]\end{aligned}$$

Energy Conservation in IMHD

- Energy conservation

$$
\begin{aligned}
0 = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + \rho U + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{j}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right) \\
+ \nabla\cdot\left[\left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + p + \rho U + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{j}|^2}{2}\right)\boldsymbol{V}\right. \\
\left. + \epsilon\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\boldsymbol{j})\boldsymbol{j} - \delta\frac{m_e}{2e^3 n^2}|\boldsymbol{j}|^2\boldsymbol{j} + \frac{\boxed{\boldsymbol{E}}\times\boldsymbol{B}}{\mu_0}\right]
\end{aligned}
$$

$$
\boxed{\boldsymbol{E}} + \boldsymbol{V}\times\boldsymbol{B} = \epsilon\frac{m_e}{e^2 n}\left[\boxed{\frac{\partial\boldsymbol{j}}{\partial t}} + \nabla\cdot(\boldsymbol{V}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{V})\right] - \delta\frac{m_e}{e^2 n}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en},
$$

This is not a correct flux form…

but total energy H is conserved!

$$
H \equiv \int\left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + \rho U + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{j}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right)d\boldsymbol{r},
$$

Classification of IMHD Models

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right)=-\nabla p+\boldsymbol{j}\times\boldsymbol{B}-\boxed{\epsilon_{\mathrm{EOM}}\frac{m_e}{e}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en}},$$

$$\begin{aligned}\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=&\boxed{\epsilon_{\mathrm{t}}\frac{m_e}{e^2n}\frac{\partial \boldsymbol{j}}{\partial t}}+\boxed{\epsilon_{\mathrm{ad}}\frac{m_e}{e^2n}(\boldsymbol{V}\cdot\nabla)\boldsymbol{j}}+\boxed{\epsilon_{\mathrm{cp}}\frac{m_e}{e^2n}\boldsymbol{j}(\nabla\cdot\boldsymbol{V})}\\&+\boxed{\epsilon_{\mathrm{M}}\frac{m_e}{e^2n}(\boldsymbol{j}\cdot\nabla)\boldsymbol{V}}-\delta\frac{m_e}{e^2n}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en},\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\boldsymbol{V}|^2+\rho U+\boxed{\epsilon_{\mathrm{t}}\frac{m_e}{e^2n}\frac{|\boldsymbol{j}|^2}{2}}+\frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\\&+\nabla\cdot\left[\left(\frac{1}{2}\rho|\boldsymbol{V}|^2+p+\rho U+\boxed{\epsilon_{\mathrm{ad}}\frac{m_e}{e^2n}\frac{|\boldsymbol{j}|^2}{2}}\right)\boldsymbol{V}\right.\\&\left.\qquad+\boxed{\epsilon_{\mathrm{M}}\frac{m_e}{e^2n}(\boldsymbol{V}\cdot\boldsymbol{j})\boldsymbol{j}}-\delta\frac{m_e}{e^3n^2}\frac{|\boldsymbol{j}|^2}{2}\boldsymbol{j}+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]\\&=\boxed{\begin{aligned}&(\epsilon_{\mathrm{t}}-\epsilon_{\mathrm{ad}})\frac{m_e}{e^2n}\frac{|\boldsymbol{j}|^2}{2}\frac{\nabla\cdot(n\boldsymbol{V})}{n}+(\epsilon_{\mathrm{ad}}-\epsilon_{\mathrm{cp}})\frac{m_e}{e^2n}|\boldsymbol{j}|^2(\nabla\cdot\boldsymbol{V})\\&+(\epsilon_{\mathrm{M}}-\epsilon_{\mathrm{EOM}})\frac{m_e}{e}\boldsymbol{V}\cdot\left\{(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en}\right\}.\end{aligned}}\end{aligned}$$

Classification of IMHD Models

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right)=-\nabla p+\boldsymbol{j}\times\boldsymbol{B}-\boxed{\epsilon_{\mathrm{EOM}}\frac{m_e}{e}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en}},$$

$$\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=\boxed{\epsilon_{\mathrm{t}}\frac{m_e}{e^2n}\frac{\partial\boldsymbol{j}}{\partial t}}+\boxed{\epsilon_{\mathrm{ad}}\frac{m_e}{e^2n}(\boldsymbol{V}\cdot\nabla)\boldsymbol{j}}+\boxed{\epsilon_{\mathrm{cp}}\frac{m_e}{e^2n}\boldsymbol{j}(\nabla\cdot\boldsymbol{V})}$$

$$+\boxed{\epsilon_{\mathrm{M}}\frac{m_e}{e^2n}(\boldsymbol{j}\cdot\nabla)\boldsymbol{V}}-\delta\frac{m_e}{e^2n}(\boldsymbol{j}\cdot\nabla)\frac{\boldsymbol{j}}{en},$$

ϵ_{t}	ϵ_{ad}	ϵ_{cp}	ϵ_{M}	Ohm's law $\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=$	ϵ_{EOM}	Conserved?
Compressible fluid						
1	1	1	1	$\frac{m_e}{e^2n}\left(\frac{\partial\boldsymbol{j}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{j}+\boldsymbol{j}\boldsymbol{V})\right)$	1	OK!
1	1	1		$\frac{m_e}{e^2n}\left(\frac{\partial\boldsymbol{j}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{j})\right)$		OK!
1						
1						
1						

The epsilon term in the momentum equation is important!!

Ohm’s Law in Previous Studies

Vasyliunas (1975)

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = \eta\mathbf{J} + \frac{m_e}{ne^2}\left[\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{JV} + \mathbf{VJ})\right]$$

$$-\frac{1}{ne}\nabla\cdot\mathbf{P}^{(e)} + \frac{1}{}$$

Fitzpatrick (2001)

$$\mathrm{E} + \mathrm{V}\times\mathrm{B} \simeq \frac{\mathrm{F_U}}{\mathrm{n}} + \mathrm{j}$$

$$- \frac{\mathrm{m_e}}{\mathrm{n^2 e^3}}(\mathrm{j}\cdot\nabla)\mathrm{j}$$

Watson

$$\mathrm{R} \quad \mathrm{J} \quad - \nabla\cdot\mathrm{p}_e) + \frac{m_e}{ne^2}\left(\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{vJ} + \mathbf{Jv})\right)$$

Bhattacharjee et a

$$\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = \eta\mathbf{J} + \frac{4\pi}{\omega_{pe}^2}\frac{D\mathbf{J}}{Dt} - \frac{\nabla p}{ne} + \frac{\mathbf{J}\times\mathbf{B}}{nec}$$

Sh)

$$\frac{4\pi}{\omega_{pe}^2}\frac{d\mathrm{J}}{dt} = \mathrm{E} + \frac{1}{c}\mathrm{v}\times\mathrm{B} - \frac{1}{nec}\mathrm{J}\times\mathrm{B} + \frac{1}{ne}\nabla\cdot\vec{\vec{\mathrm{P}}}_e - \eta\mathrm{J},$$

Total energy is not conserved!!

There is no $\frac{m_e}{e}(j\cdot\nabla)\frac{j}{en}$ term in the momentum equation!

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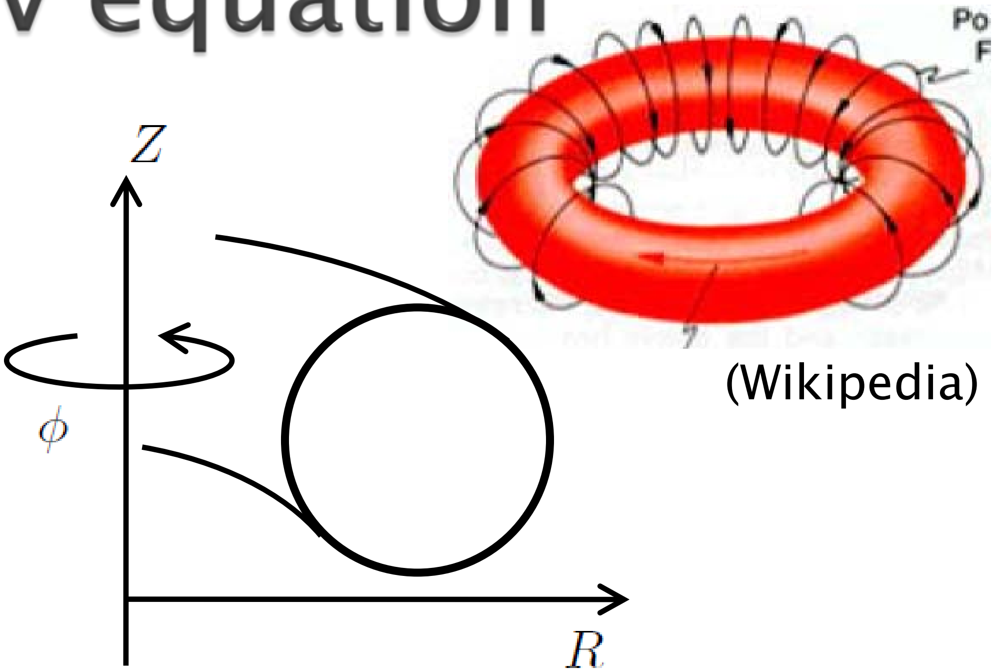
Some Equilibrium States

- With no flow
  1. Grad–Shafranov equation

- With flow (incompressible)
  1. $V \propto j$
  2. Beltrami–“Jeltrami” flow

Grad–Shafranov equation



(Wikipedia)

- Governing equation (no flow)

$$\mathbf{0} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

- Axisymmetric solution

$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + \frac{1}{R} \boldsymbol{e}_\phi \times \nabla \psi$$

⇒

$$p \equiv p(\psi), \quad RB_\phi \equiv F(\psi)$$

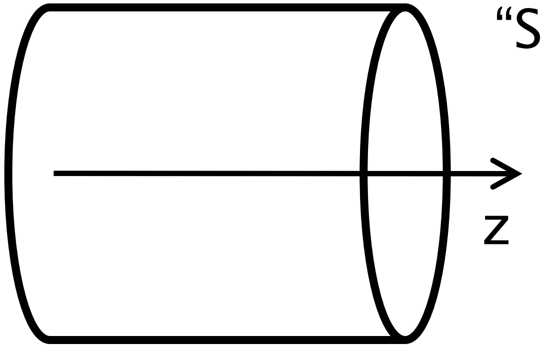
$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}$$

: Grad–Shafranov equation

$$\Delta^* \equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2}$$

What occurs with the electron inertia?

G–S in “Straight Torus”



“Straight torus” = cylinder; torus with no curvature

$$\mathbf{0} = -\frac{1}{m_e n}\nabla p + \frac{1}{m_e n}\boldsymbol{j}\times\boldsymbol{B} - \epsilon\left(\frac{\boldsymbol{j}}{en}\cdot\nabla\right)\frac{\boldsymbol{j}}{en},$$

$$\mathbf{0} = \nabla\times\boldsymbol{E} = -\delta\frac{m_e}{e}\nabla\times\left[\left(\frac{\boldsymbol{j}}{en}\cdot\nabla\right)\frac{\boldsymbol{j}}{en}\right],$$

- “Axisymmetric” solution = z independent solution

$$\boldsymbol{B} = B_z\boldsymbol{e}_z + \boldsymbol{e}_z\times\nabla\psi,$$

- When $\epsilon,\ \delta\to 0$, then we find that $p\equiv p(\psi)$ and $B_z\equiv F(\psi)$,

  and obtain $$\nabla_\perp^2\psi = -\mu_0\frac{dp}{d\psi} - F\frac{dF}{d\psi}$$: G–S in “straight torus”

What occurs when epsilon and delta are finite?

Modified G-S in “Straight Torus”

- If the plasma is barotropic, i.e., $n = n(p)$,

$$\nabla_\perp^2 \psi = -\frac{1}{1-\epsilon d_e^2 F'^2}(\mu_0 p' + FF') - \frac{\epsilon d_e^2 F'^2}{1-\epsilon d_e^2 F'^2}\frac{K'}{2},$$

Modified Grad-Shafranov equation in “straight torus”

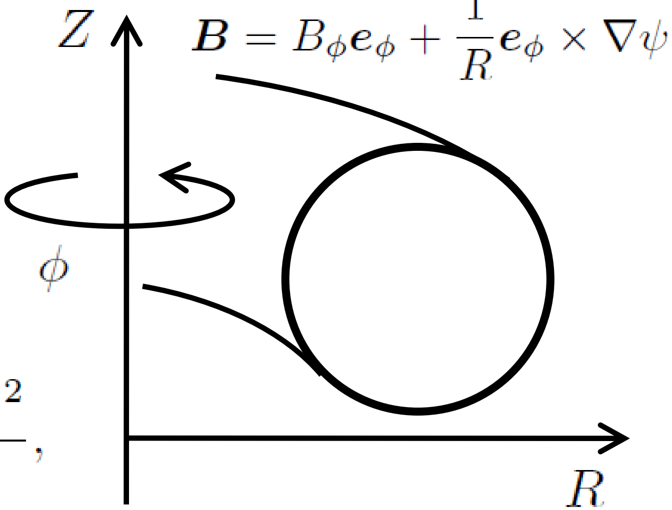
$$p \equiv p(\psi) \qquad |\nabla_\perp \psi|^2 \equiv K(\psi),$$

$$B_z \equiv F(\psi)$$

$$d_e^2 \equiv \frac{m_e}{\mu_0 e^2 n}$$

$$n = n(\psi)$$

Modified G-S equation in torus



$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + \frac{1}{R}\boldsymbol{e}_\phi \times \nabla\psi$$

- Incompressible plasma

$$\mathbf{0} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - \epsilon \frac{m_e}{e^2 n_0} (\boldsymbol{j} \cdot \nabla)\boldsymbol{j},$$

$$\Rightarrow \left\{ \begin{aligned} &\psi - \tilde{\epsilon}\Delta^*\psi \equiv L(RB_\phi), \\ &M(RB_\phi) \equiv \mu_0 p + \tilde{\epsilon}\frac{(\Delta^*\psi)^2}{2R^2} + \tilde{\epsilon}\frac{|\nabla_\perp(RB_\phi)|^2}{2R^2}, \\ &0 = R^2 \frac{dM}{d(RB_\phi)} + \frac{dL}{d(RB_\phi)}\Delta^*\psi - \tilde{\epsilon}\Delta^*(RB_\phi) + RB_\phi. \end{aligned} \right.$$

$$\tilde{\epsilon} \equiv \epsilon\, d_{e0}^2 = \epsilon\, \frac{m_e}{\mu_0 e^2 n_0} \qquad \Delta^* \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial Z^2}$$

$$\mathbf{0} = \nabla \times \boldsymbol{E} = -\delta \frac{m_e}{e^3 n_0^2} \nabla \times [(\boldsymbol{j} \cdot \nabla)\boldsymbol{j}], \quad \Rightarrow \quad ??$$

I did not get
any useful constraints…

Summary of G-S in IMHD

	Straight torus	Torus
Incompressible	3 constraints obtained	3 constraints obtained
	Modified G-S obtained	??
Compressible	If barotropic, 3 (incomplete) constraints obtained.	?
	If barotropic, modified GS obtained.	???

Epsilon term is considered

Epsilon and delta term are considered

Some Equilibrium States

- With no flow
  1. Grad–Shafranov equation

- With flow (incompressible)
  1. $\boldsymbol{V} \propto \boldsymbol{j}$
  2. Beltrami–“Jeltrami” flow

$\boldsymbol{V} \propto \boldsymbol{j}$

- Governing equations (incompressible IMHD)

$$\mathbf{0} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - \epsilon \frac{m_e}{e^2 n_0} (\boldsymbol{j} \cdot \nabla) \boldsymbol{j} - \rho_0 (\boldsymbol{V} \cdot \nabla) \boldsymbol{V},$$

$$\nabla \Phi = \boldsymbol{E} = -\boldsymbol{V} \times \boldsymbol{B} + \epsilon \frac{m_e}{e^2 n_0} [(\boldsymbol{V} \cdot \nabla) \boldsymbol{j} + (\boldsymbol{j} \cdot \nabla) \boldsymbol{V}] - \delta \frac{m_e}{e^3 n_0^2} (\boldsymbol{j} \cdot \nabla) \boldsymbol{j},$$

- Assuming $\boldsymbol{V} = C\boldsymbol{j},$

$$0 = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - \left(\rho_0 C^2 + \epsilon \frac{m_e}{e^2 n_0} \right) (\boldsymbol{j} \cdot \nabla) \boldsymbol{j},$$

$$0 = \nabla \Phi + C \boldsymbol{j} \times \boldsymbol{B} - \left(2C\epsilon \frac{m_e}{e^2 n_0} - \delta \frac{m_e}{e^3 n_0^2} \right) (\boldsymbol{j} \cdot \nabla) \boldsymbol{j}$$

quite similar!

- If $\delta = 0,$

$$C = \pm \sqrt{\epsilon \frac{m_e}{e^2 n_0 \rho_0}}$$

$$\Phi \equiv -Cp,$$

$$\boxed{0 = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - 2C^2 (\boldsymbol{j} \cdot \nabla) \boldsymbol{j},}$$

Some Equilibrium States

- With no flow
  1. Grad–Shafranov equation

- With flow (incompressible)
  1. $\boldsymbol{V} \propto \boldsymbol{j}$
  2. Beltrami–“Jeltrami” flow

Beltrami–“Jeltrami” Flow

$\nabla \times \boldsymbol{V} = \lambda \boldsymbol{V}$: Beltrami flow

$\nabla \times \boldsymbol{j} = \mu \boldsymbol{j}$: “Jeltrami” current

$$\Rightarrow \quad (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \nabla \left(\frac{|\boldsymbol{V}|^2}{2} \right) - \cancel{\boldsymbol{V} \times (\nabla \times \boldsymbol{V})}$$

$$\boxed{\begin{aligned} \nabla \tilde{p} &= \boldsymbol{j} \times \boldsymbol{B}, \\ \nabla \tilde{\Phi} &= -\boldsymbol{V} \times \boldsymbol{B} + \epsilon \frac{m_e}{e^2 n_0} (\mu - \lambda) \boldsymbol{j} \times \boldsymbol{V} \end{aligned}}$$

$$\tilde{p} \equiv p + \rho_0 \frac{|\boldsymbol{V}|^2}{2} + \epsilon \frac{m_e}{e^2 n_0} \frac{|\boldsymbol{j}|^2}{2},$$

$$\tilde{\Phi} \equiv \Phi + \epsilon \frac{m_e}{e^2 n_0} (\boldsymbol{V} \cdot \boldsymbol{j}) + \delta \frac{m_e}{e^3 n_0^2} \frac{|\boldsymbol{j}|^2}{2},$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = \frac{\mu_0}{\mu} \nabla \times \boldsymbol{j}, \quad \Rightarrow \quad \boldsymbol{B} = \frac{\mu_0}{\mu} \boldsymbol{j} + \nabla \chi,$$

If $\mu = \lambda, \quad \text{and} \quad \chi \equiv 0,$

$$\boxed{\begin{aligned} \nabla \tilde{p} &= 0, \\ \nabla \tilde{\Phi} &= \frac{\mu_0}{\mu} \boldsymbol{j} \times \boldsymbol{V} \end{aligned}}$$

← similar to Bernoulli’s equation

Conclusions

- Modified Alfven wave in Inertial MHD is dispersive.
- The epsilon term in the momentum equation is important in terms of energy conservation.
- Modified Grad-Shafranov equation is obtained in "straight torus." In the real torus, we obtained only some constraints.
- Governing equations of equilibrium states with flow can be simplified in $V \propto j$ and in Beltrami-"Jeltrami" flow.

Future Works

- Physical interpretation of equilibrium state with delta term in modified Grad-Shafranov equation with curvature
- Studying the stability with the effect of epsilon and delta term
- Shocks
- Magnetic reconnection

Menu

- Research Project in GFD Program 2011
- Lectures in GFD Program 2011
- Life at GFD

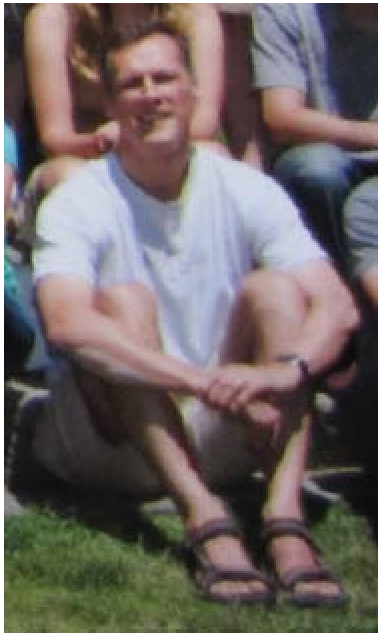
Topic in GFD Program 2011

"Shear Turbulence: Onset and Structure"

(1988)

Fabian Waleffe
(University of Wisconsin, Madison)

(2011)



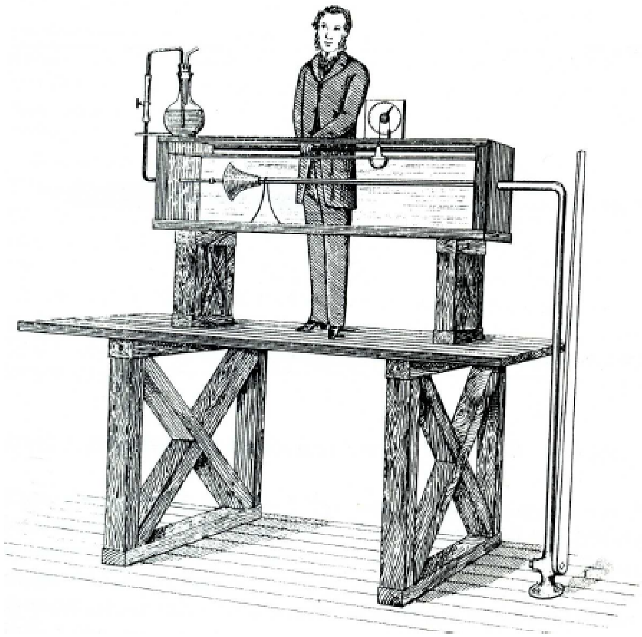
(2011)

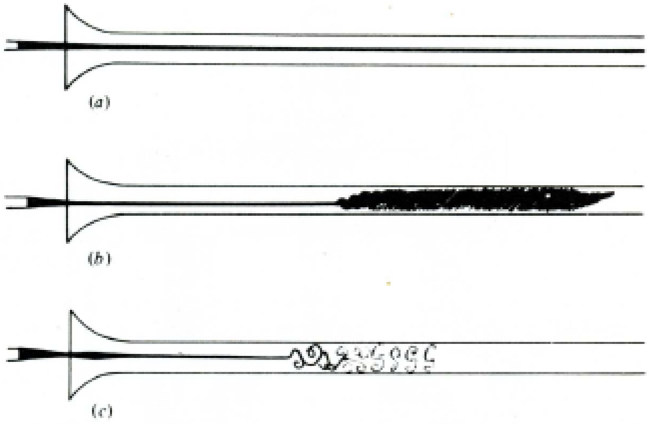
(1990)

(1995)

Richard Kerswell
(Bristol University)

Turbulent Onset in Shear Flows





Canonical
Laminar flow

"Turbulent"
State

Transition occurs suddenly,
noise-dependently and
dramatically!

Linear Stability

Canonical Flow	Critical Reynolds number
Plane Couette	∞
Plane Poiseuille	5772
Hagen–Poiseuille (Pipe)	∞?

"Turbulent" State

Transition can occur at lower Reynolds number!

Why? What occurs?

Waleffe's Lectures

1. General Introduction and Overview.
2. Viscous derivation of classic inviscid stability results for shear flows. Viscous instability.
3. Diffusion and damping in shear flows: a truly singular limit. Critical layers.
4. Origin and survival of 3D-ality.
5. Instability of streaky flows. Asymptotics of self-sustaining process.
6. Spatio-temporal complexity. Spots, puffs and slugs, snakes and spirals.

Summary

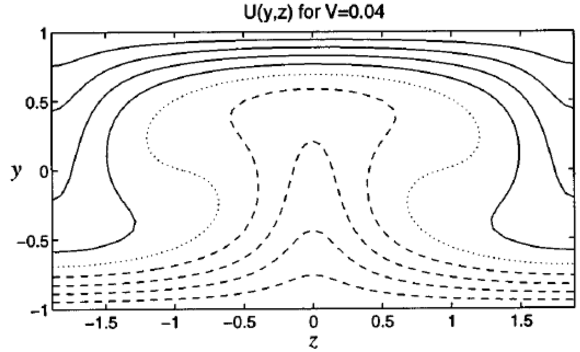
- Transition threshold i.e. the power law of the amplitude against Re

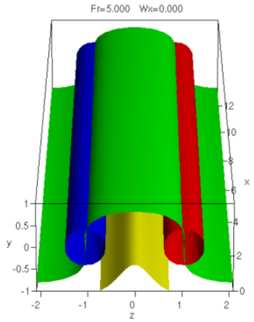
- Structure (horseshoe vortex etc.)
- Linear and energy stabilities
  - → Squire's theorem (linear) vs. streamwise roll (energy)

- How to sustain the turbulent state
  - → the feedback mechanism to the roll pattern
- Self-Sustaining Process (SSP) and SSP method
  - → finding the exact coherent structures

- The boundary of the laminar and turbulent states in phase space

Self-Sustaining Process (SSP)





Streaks

O(1)

instability of U(y,z)

Streak wave mode (3D)

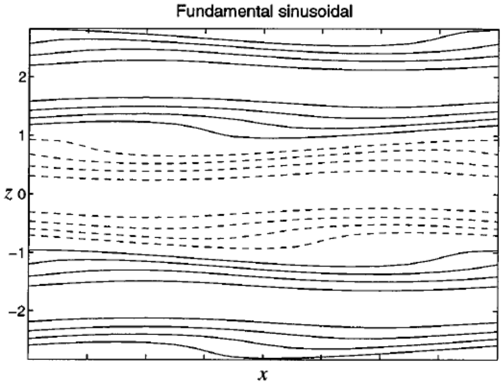
O(1/R)

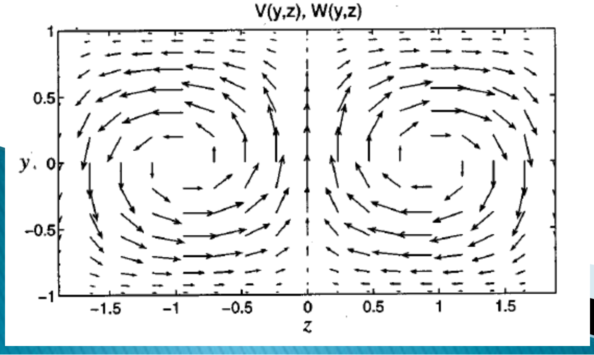
nonlinear self–interaction

Streamwise Rolls

O(1/R)

advection of mean shear





Waleffe (1997)

4th Order ODE

$M(t)$ = amp of mean shear $\overline{U}(y)\hat{\mathbf{x}}$

$U(t)$ = amp of streaks $u(y,z)\hat{\mathbf{x}}$

$V(t)$ = amp of streamwise rolls $v(y,z)\hat{\mathbf{y}} + w(y,z)\hat{\mathbf{z}}$

$W(t)$ = amp of streak eigenmode $\mathbf{u}(x,y,z)$

Captures basic features of SSP

$$
\begin{array}{rcl|c|c|c|c|}
\left(\frac{d}{dt} + \frac{\kappa_m^2}{R}\right) M & = & \frac{\kappa_m^2}{R} & -\sigma_u UV & & & +\sigma_m W^2 \\
\left(\frac{d}{dt} + \frac{\kappa_u^2}{R}\right) U & = & & \sigma_u MV & -\sigma_w W^2 & & \\
\left(\frac{d}{dt} + \frac{\kappa_v^2}{R}\right) V & = & & & & \sigma_v W^2 & \\
\left(\frac{d}{dt} + \frac{\kappa_w^2}{R}\right) W & = & & & \sigma_w UW & -\sigma_v VW & -\sigma_m MW
\end{array}
$$

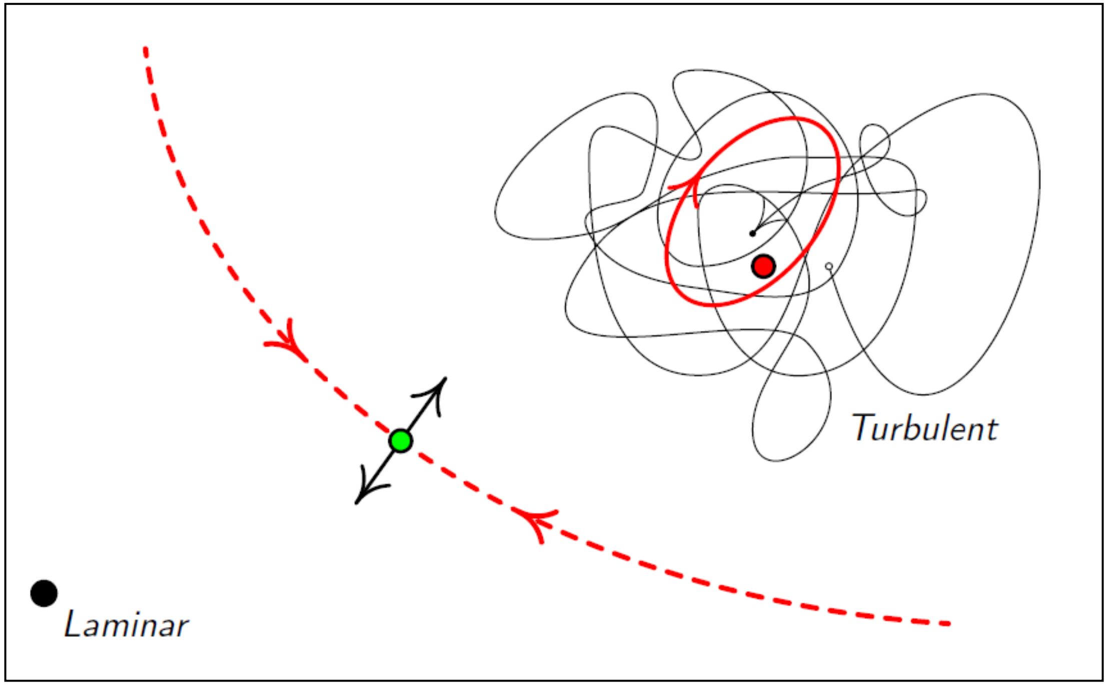
Rolls V redistribute streamwise momentum M → U

Streak U unstable to x-mode W

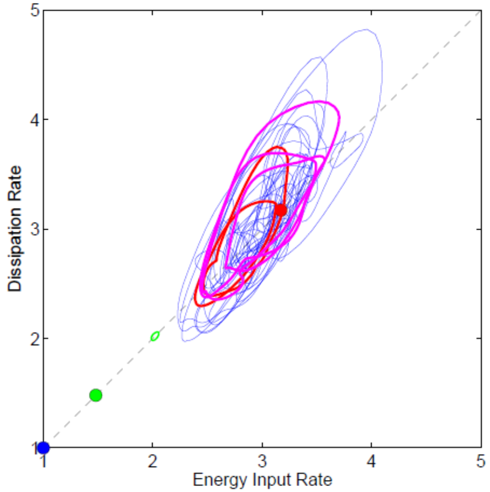
nonlinear interaction of W feeds back on rolls V

Mean M shears W

Unstable Coherent States!



PCF data (*R* = 400)



Periodic solutions in HKW (1.14, 1.67) by *Viswanath, JFM 2007* & *Gibson (TBA)*

Kerswell's Lectures

1. Transition scenarios: normality vs non-normality.
2. Edge tracking – walking the tightrope.
3. Triggering transition efficiently.
4. Turbulence: transient or sustained?

Summary

- Supercritical or subcritical scenarios
  ← Normal or Non-normal linear operator

- Finding nonlinear solutions
  (e.g. Nagata's solution)

- Edge tracking

- Finding Minimal Seeds

- Is the turbulent state is a transient state or sustained one? ← puff, localized pattern

Minimal Seed

“Minimal seed”: the I.C. of smallest energy
which can trigger transition to “turbulence”

→ Can we identify the minimal seed by looking for an I.C. which experiences largest growth?

Energy growth rate: $$G(T) \equiv \max_{\mathrm{u}_0(\mathrm{x}),\ \nabla\cdot\mathrm{u}_0=0} \frac{\left\langle \frac{1}{2}|\mathrm{u}(\mathrm{x},T)|^2 \right\rangle}{\left\langle \frac{1}{2}|\mathrm{u}_0(\mathrm{x})|^2 \right\rangle},$$

- Linear transient growth
  Matrix-based → SVD
  Matrix-free → Variational principle (Euler-Lagrange eq.)
  this method can be used to study
  the nonlinear transient growth!

Variational Method

- Linearized Navier-Stokes eq.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_{\mathrm{lam}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_{\mathrm{lam}} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u},$$

- Growth rate

$$\begin{aligned} G \equiv & \left\langle \frac{1}{2}|\mathbf{u}(\mathbf{x}, T)|^2 \right\rangle + \lambda \left\{ \left\langle \frac{1}{2}|\mathbf{u}(\mathbf{x}, 0)|^2 \right\rangle - 1 \right\} \\ & + \int_0^T \left\langle \mathbf{v}(\mathbf{x}, t) \cdot \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_{\mathrm{lam}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_{\mathrm{lam}} + \nabla p - \frac{1}{Re}\nabla^2 \mathbf{u} \right\} \right\rangle dt \\ & + \int_0^T \langle \pi(\mathbf{x}, t) \nabla \cdot \mathbf{u} \rangle \, dt \end{aligned}$$

- Euler-Lagrange eqns.

$$\frac{\delta G}{\delta \mathbf{u}(\mathbf{x}, T)} = 0 \quad \Rightarrow \quad \mathbf{u}(\mathbf{x}, T) + \mathbf{v}(\mathbf{x}, T) = 0$$

$$\frac{\delta G}{\delta \mathbf{u}(\mathbf{x}, 0)} = 0 \quad \Rightarrow \quad \lambda \mathbf{u}(\mathbf{x}, 0) - \mathbf{v}(\mathbf{x}, 0) = 0$$

$$\frac{\delta G}{\delta \mathbf{u}} = 0 \quad \Rightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u}_{\mathrm{lam}} \cdot \nabla)\mathbf{v} - \mathbf{v} \cdot (\nabla \mathbf{u}_{\mathrm{lam}})^{\mathrm{T}} + \nabla \pi + \frac{1}{Re}\nabla^2 \mathbf{v} = 0,$$

Dual Linearized Navier-Stokes eqn.

Diagram of Iterative Method

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_{\mathrm{lam}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_{\mathrm{lam}} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}$$

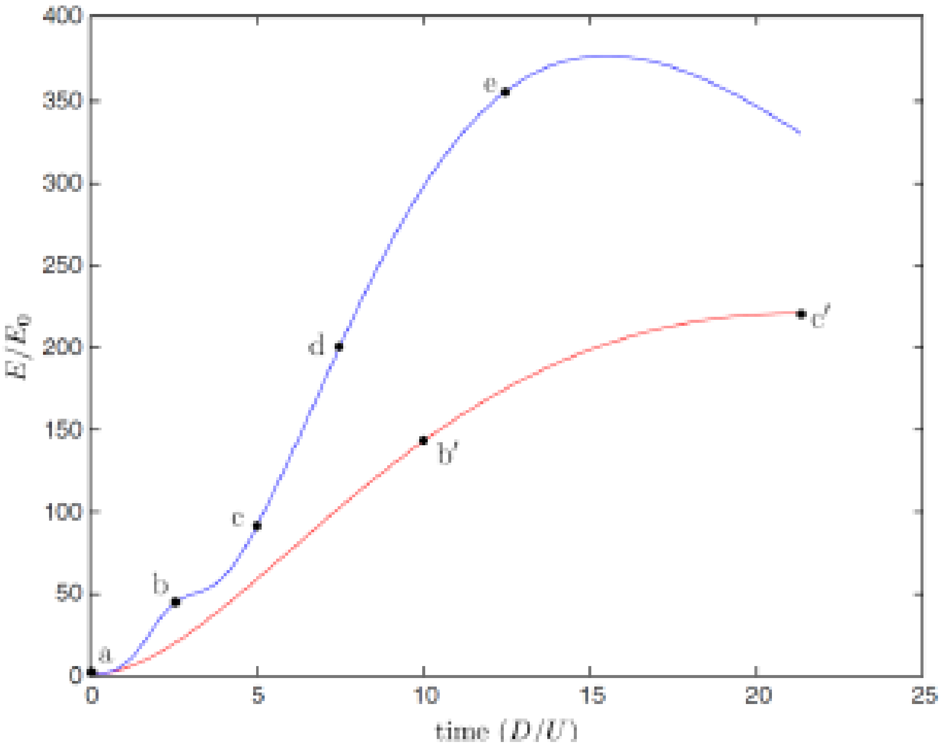
$$\begin{tikzcd}
 & \mathbf{u}(\mathbf{x},0) \arrow[rr, "\text{Step.1}", "\begin{array}{l}\text{Linearized Navier-Stokes equation}\\ \text{Incompressibility}\\ \text{Boundary condition}\end{array}"'] & & \mathbf{u}(\mathbf{x},T) \arrow[dd, "\text{Step.2}"'] & \\
\dfrac{\delta G}{\delta \mathbf{u}(\mathbf{x},0)} = 0 & & & & \begin{array}{c}\dfrac{\delta G}{\delta \mathbf{u}(\mathbf{x},T)} = 0 \\ \mathbf{u}(\mathbf{x},T) + \mathbf{v}(\mathbf{x},T) = 0\end{array} \\
 & \mathbf{v}(\mathbf{x},0) \arrow[uu, "\text{Step.4}"'] & & \mathbf{v}(\mathbf{x},T) \arrow[ll, "\text{Step.3}"', "\begin{array}{l}\text{Dual linearized Navier-Stokes equation}\\ \text{Incompressibility}\\ \text{Boundary condition}\end{array}"] & \\
\begin{aligned} \mathbf{u}^{(n+1)} &= \mathbf{u}^{(n)} + \epsilon \left[\frac{\delta G}{\delta \mathbf{u}(\mathbf{x},0)}\right]^{(n)} \\ &= \mathbf{u}^{(n)} + \epsilon \left(\lambda \mathbf{u}(\mathbf{x},0)^{(n)} - \mathbf{v}(\mathbf{x},0)^{(n)}\right) \end{aligned} \arrow[uu] & & & &
\end{tikzcd}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u}_{\mathrm{lam}} \cdot \nabla)\mathbf{v} - \mathbf{v} \cdot (\nabla \mathbf{u}_{\mathrm{lam}})^{\mathrm{T}} + \nabla \pi + \frac{1}{Re}\nabla^2 \mathbf{v} = 0,$$

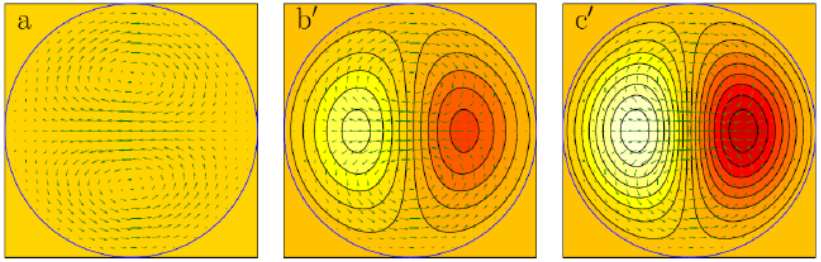
This method is easily extendable to the nonlinear problem!

Results

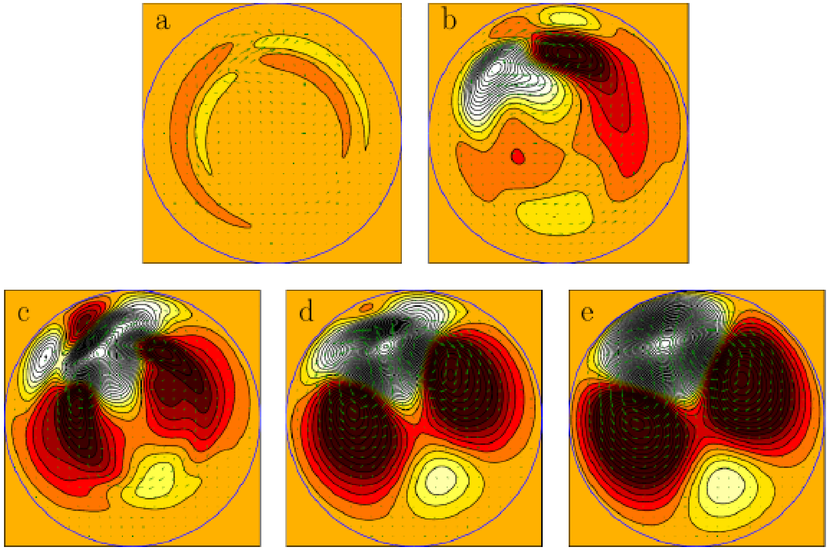
Pringle and Kerswell (2010)

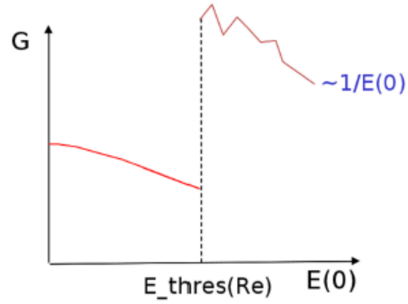


- Linear



- Nonlinear





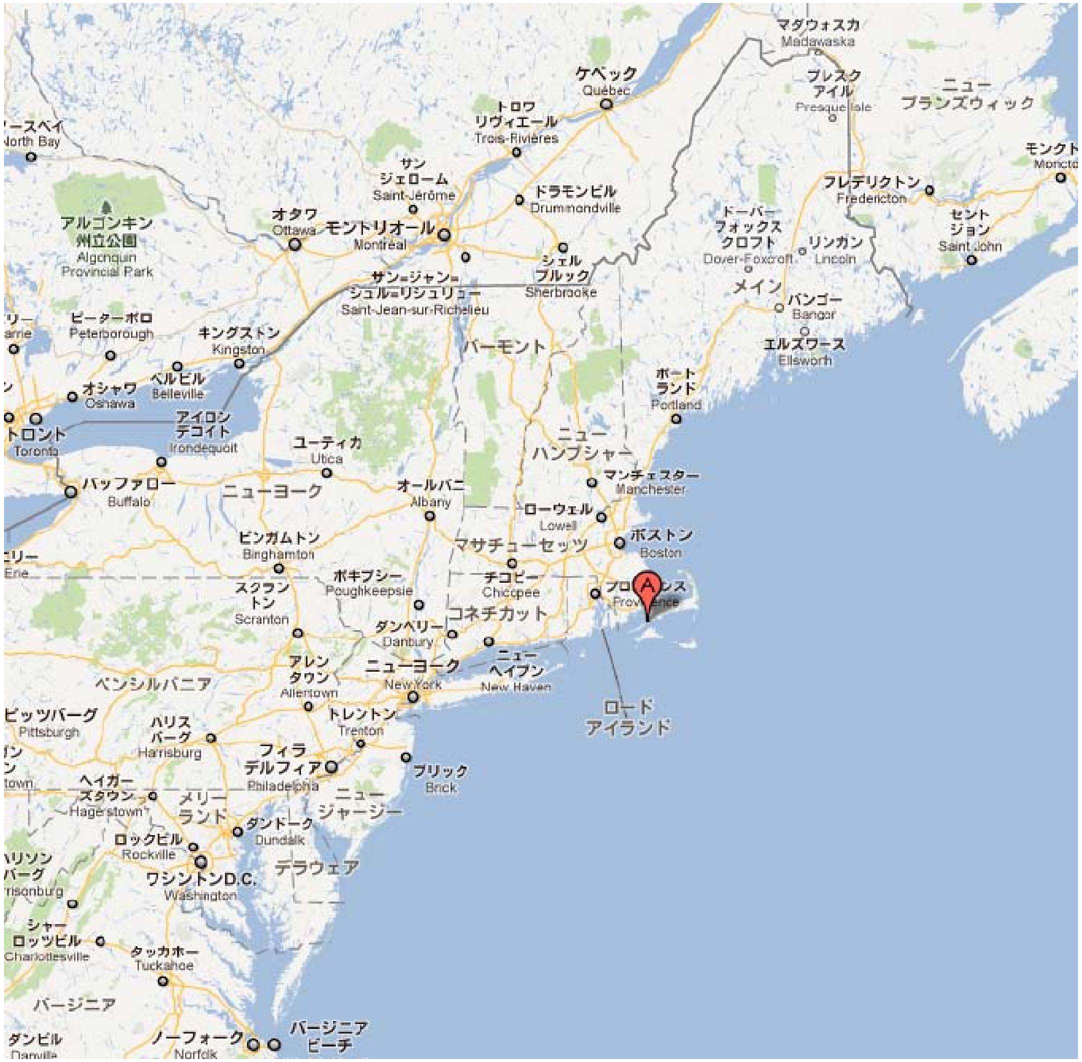
Schmid and Henningson (1994)

Menu

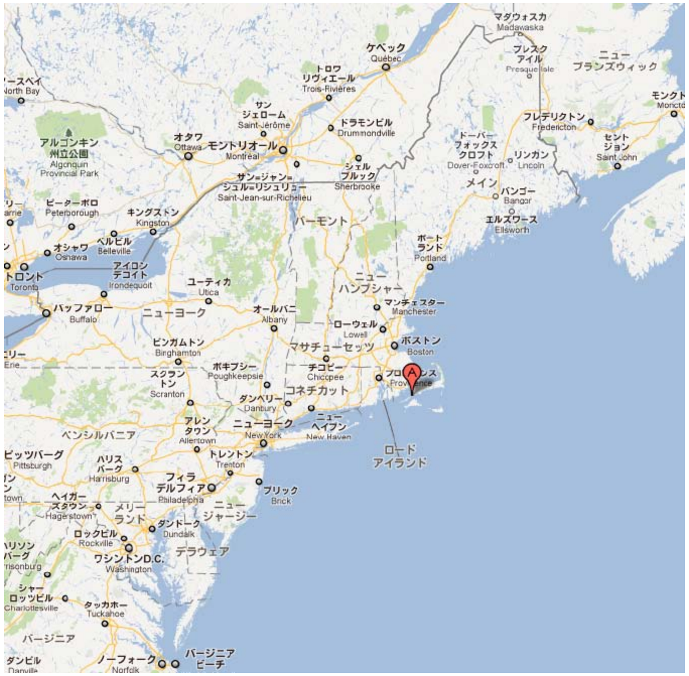
- Lectures in GFD Program 2011
- Research Project in GFD Program 2011
- Life at GFD

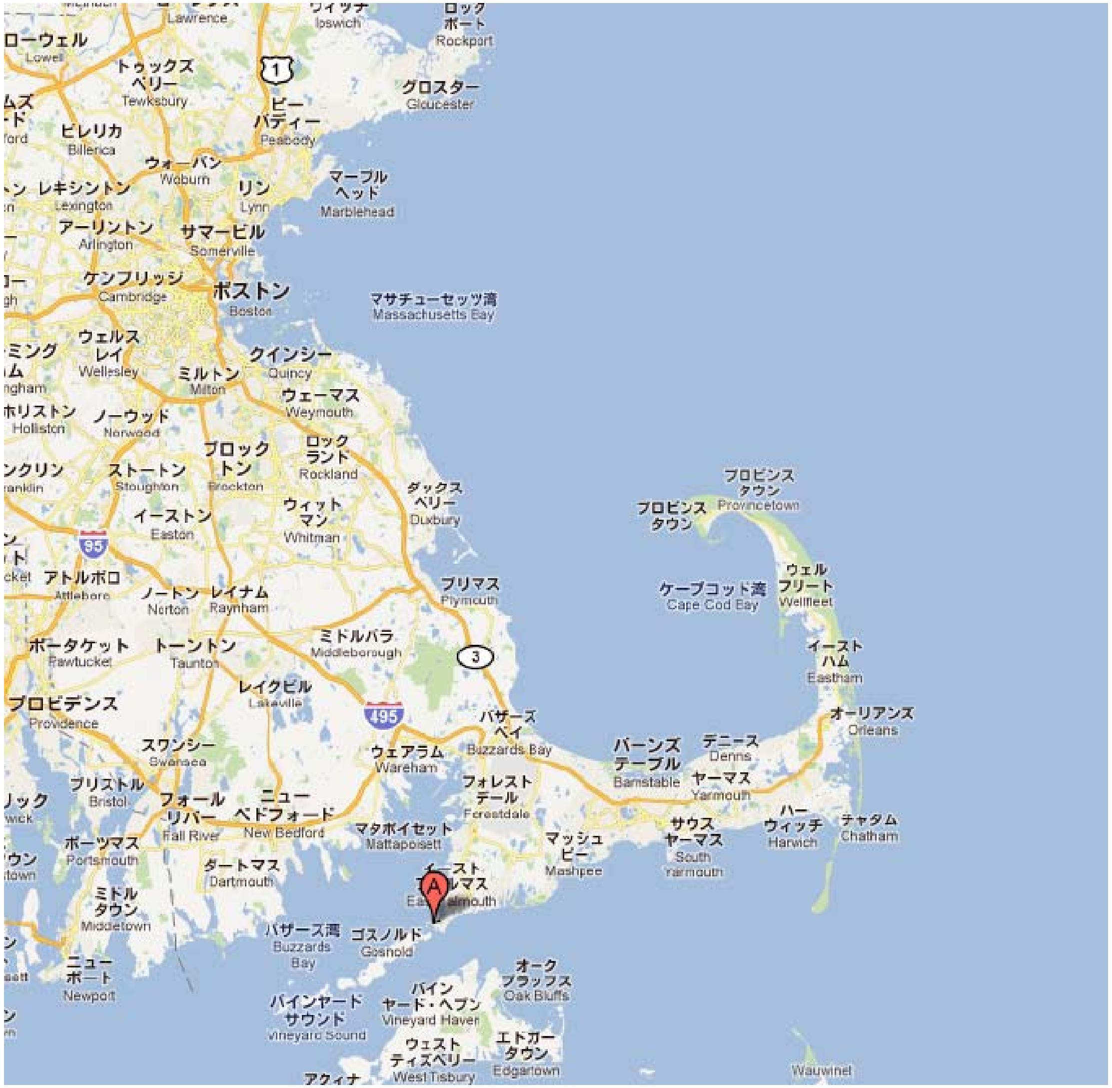
Woods Hole Oceanographic Institution

Woods Hole Oceanographic Institution



Woods Hole Oceanographic Institution





Fellows

Zhan Wang 王展
(Univ. of Wisconsin, Madison)
Chinese (28)

Samuel Potter
(Princeton Univ.)
American (29?)

Andrew Crosby
(Univ. of Cambridge)
British (26)

Adele Morrison
(Australian National Univ.)
Australian (29)

John Platt
(Harvard Univ.)
British (24)

Lindsey Ritchie (Corson)
(Univ. of Strathclyde)
British (27?)

Keiji Kimura
(Kyoto Univ.)
Japanese (25)

Matthew Chantry
(Univ. of Bristol)
British (23)

Chao Ma 馬超
(Univ. of Colorado, Boulder)
Chinese (28)

Giulio Mariotti
(Boston Univ.)
Italian (27)

Martin Hoecker-Martinez
(Oregon State Univ.)
American (32)

Fellows

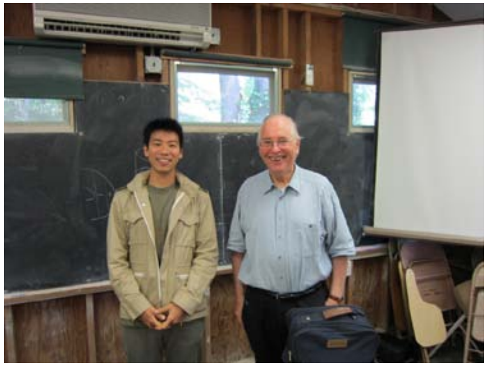
British:	4
American:	2
Chinese:	2
Australian:	1
Italian:	1
Japanese:	1

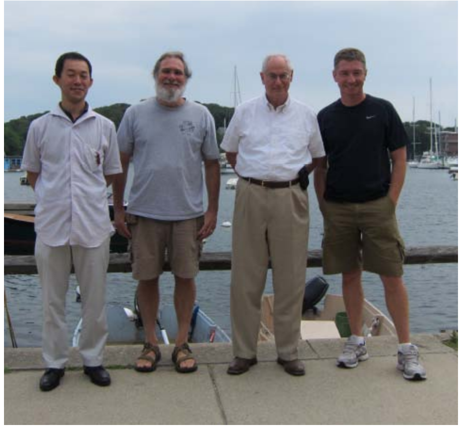
Univ.

in USA:	6
in British:	3
in Australia:	1
in Japan:	1

Talks

- Genta Kawahara, “Structures of low-Reynolds number turbulence in a rectangular duct”
- Friedrich Busse, “Generation of magnetic fields by convection in rotating spherical fluid shells”
- Jesse Ausubel, “Self-sinking capsules to investigate Earth’s interior and dispose of radioactive waste”
- Tomoaki Itano, “Coherent vortices in plane Couette flow - bifurcation, symmetry and visualization”
- Predrag Cvitanovic, “What Phil Morrison would not teach us: how to reduce the symmetry of pipe flows”





Fellows' Research Projects

Martin Hoecker-Martinez: Constraints on low order models: the cost of simplicity

Keiji Kimura: A One-fluid MHD Model with Electron Inertial

Matthew Chantry: Traversing the edge: how turbulence decays

Giulio Mariotti: A low dimensional model for shear turbulence in Plane Poiseuille Flow: an example to understand the edge

Adele Morrison: Upstream basin circulation of rotating, hydraulically controlled flows

Samuel Potter: Islands in locally-forced basin circulations

Zhan Wang: Tow-layer viscous fluid in an inclined closed tube: Kelvin-Helmholtz instability

Andrew Crosby: Chaotic interaction of vortex patches with boundaries

Chao Ma: On Brownian motion in a Fluid with a Plane Boundary

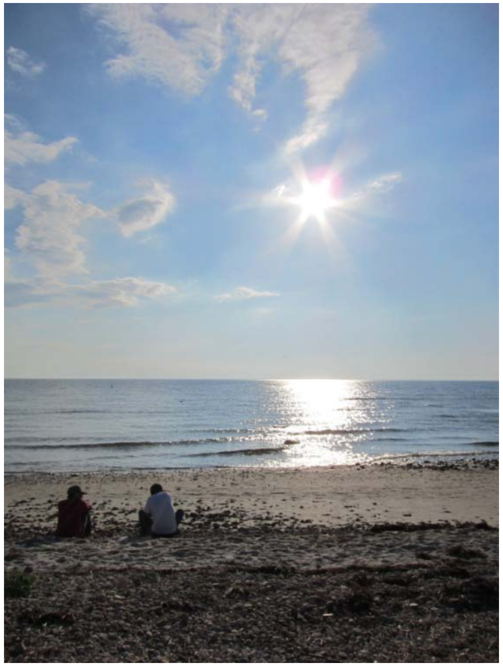
John Platt: Localized Solutions for Plane Couette flow: a continuation method study

Lindsey Ritchie: Ascending the ridge: Maximizing the heat flux in steady porous medium convection

Softball

5勝3敗!!

Others

得たもの，失ったもの

- 得たもの

  生活力，料理の技術

  アメリカ人の人生の楽しみ方

  スポーツの重要性

  度胸（英語力）

  Shear Turbulence の知識

  MHD の知識

- 失ったもの

  体重 (–5kg)

Acknowledgements

林先生をはじめ，北大・神戸大GCOE, CPS

関係者の皆様に深く感謝申し上げます

(1984)